

hep-th/0505233  
May 2005

# Re-Identifying the Hagedorn Transition

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## Abstract

The Hagedorn transition in string theory is normally associated with an exponentially rising density of states, or equivalently with the existence of a thermal string winding mode which becomes tachyonic above a specific temperature. However, the details of the Hagedorn transition turn out to depend critically on the precise manner in which a zero-temperature string theory is extrapolated to finite temperature. In this paper, we argue that for broad classes of closed string theories, the traditional Hagedorn transition is completely absent when the correct extrapolation is used. However, we also argue that there is an alternative “re-identified” Hagedorn transition which is triggered by the thermal winding excitations of a different, “effective” tachyonic string ground state. These arguments allow us to re-identify the Hagedorn temperature for heterotic strings. Moreover, we find that all tachyon-free closed string models in ten dimensions share the same (revised) Hagedorn temperature, resulting in a universal Hagedorn temperature for both Type II and heterotic strings. We also comment on the possibility of thermal spin-statistics violations at the Planck scale.

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# 1 Motivation, overview, and summary of results

The Hagedorn transition is one of the central hallmarks of string thermodynamics. Originally discovered in the 1960’s through studies of hadronic resonances and the so-called “statistical bootstrap” [1, 2, 3], the Hagedorn transition is now understood to be a generic feature of any theory exhibiting a density of states which rises exponentially as a function of mass. In string theory, the number of states of a given total mass depends on the number of ways in which that mass can be partitioned amongst individual quantized mode contributions, leading to an exponentially rising density of states [4]. Thus, string theories should exhibit a Hagedorn transition [5, 6, 7, 8, 9]. Originally, it was assumed that the Hagedorn temperature is a limiting temperature at which the internal energy of the system diverges. However, later studies demonstrated that the internal energy actually remains finite at this temperature. This then suggests that the Hagedorn temperature is merely the critical temperature corresponding to a first- or second-order phase transition.

There have been many speculations concerning possible interpretations of this phase transition, including a breakdown of the string worldsheet into vortices [6] or a transition to a single long-string phase [9]. It has also been speculated that there is a dramatic loss of degrees of freedom at high temperatures [8]. Over the past two decades, studies of the Hagedorn transition have reached across the entire spectrum of modern string-theory research, including open strings and D-branes, strings with non-trivial spacetime geometries (including AdS backgrounds and *pp*-waves), strings in magnetic fields,  $\mathcal{N}=4$  strings, tensionless strings, non-critical strings, two-dimensional strings, little strings, matrix models, non-commutative theories, as well as possible cosmological implications and implications for the brane world. A brief selection of papers in many of these areas appears in Refs. [10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23]. However, with only rare exceptions, the fundamental origins of the Hagedorn transition have not been seriously investigated within the context of actual finite-temperature string model-building.

In this paper, we shall undertake a critical re-evaluation of the Hagedorn phenomenon within the context of perturbative closed string theories. As we shall show, the details of the Hagedorn transition — including its very existence — depend on the precise manner in which such zero-temperature string theories are extrapolated to finite temperature. Using very general criteria, we shall argue that when a consistent extrapolation is selected, the traditional Hagedorn transition is completely absent within a broad class of closed strings consisting of all heterotic strings and certain Type II strings in  $D < 10$ . Indeed, as we shall demonstrate, the usual Hagedorn phase transition is actually not reflected in the behavior of *any* string thermodynamic quantities; one-loop thermodynamic quantities such as the free energy, the internal energy, the entropy, and even the specific heat will remain smooth and undisturbed as a function of temperature, crossing the traditional Hagedorn temperature without so much as a ripple.

So what happened to the Hagedorn transition in such theories? As we shall argue, the answer is easy to understand in the heterotic case. As we shall review in Sect. 3, one of the standard explanations for the emergence of a Hagedorn transition [8] involves the appearance of a thermal winding-mode excitation of the string ground state which becomes tachyonic beyond a critical (Hagedorn) temperature. We find, by contrast, that there is no Hagedorn divergence in the effective potential of the heterotic string because the tachyonic string ground state on which this argument is predicated is unphysical and does not appear in the actual string spectrum when a proper finite-temperature string model is constructed. Thus, it is absent from the string partition function, and does not have any thermal excitations which could give rise to a divergence.

Let us be more precise here. There are actually two ways in which a state may fail to appear in the spectrum of a given closed finite-temperature string theory:

- First, a state may fail to satisfy the appropriate finite-temperature GSO constraints. Even if it happens to satisfy the level-matching constraints, such a state is unphysical. Such states do not appear in the string partition function, and play no role in string loop calculations.
- Alternatively, a state may satisfy the finite-temperature GSO constraints, but fail to satisfy the appropriate level-matching constraints. In other words, while modular invariance ensures that the difference between the left- and right-moving worldsheet energies will be an integer, this integer may not be zero. Such a state is merely off-shell, since its contributions appear in the string partition function. Such states cannot appear as in- or out-states, but they do contribute as internal states in string loop diagrams because they satisfy the GSO constraints appropriate for the particular string model under study.

The important point, then, is that the tachyonic heterotic string ground state on which the usual winding-mode argument is based is often in the first category — unphysical rather than merely off-shell. This means that this state does not exist, except as the mathematical ground state of the conformal field theory from which the physical string states (both on-shell and off-shell) are constructed. We shall discuss this more fully in Sect. 4. Although this state still controls the exponential divergence in the density of physical string states, it is not itself a physical object. Thus, even when the mass of this state is augmented by thermal winding contributions to become massless and level-matched, this state still fails to satisfy the finite-temperature GSO constraints. It therefore does not become physical, and cannot trigger a Hagedorn transition.

Is, then, the Hagedorn transition a spurious, unphysical effect in such theories?

In this paper, we shall argue that the answer is “no”, but that the Hagedorn transition has been misidentified. Instead, we claim that there actually *is* a physical Hagedorn transition, but at a somewhat higher temperature. Moreover, we claim that

this new Hagedorn transition is completely observable in the behavior of physical, thermodynamic quantities, appearing as actual divergences or discontinuities in these quantities as functions of temperature. Indeed, we shall show that this new phase transition, unlike the traditional Hagedorn transition, is an extremely weak transition whose order depends on the spacetime dimension.

The origins of this new Hagedorn transition can be explained in a manner identical to the explanation of the traditional Hagedorn transition. Specifically, we shall show in Sect. 5 that for heterotic strings there is an off-shell (but otherwise physical) tachyon which is *not* the string ground state, but which generically appears in all finite-temperature string models. When augmented with thermal winding contributions, this state can become massless and on-shell. This then triggers a true, physical divergence (or discontinuity) in the behavior of thermodynamic quantities. We claim that for the purposes of calculating thermodynamic quantities, it is *this* state which functions as the “effective” ground state: it is this state, and not the actual string ground state, that is responsible for triggering the Hagedorn transition. Since this effective string ground state is less tachyonic than the actual string ground state, the corresponding Hagedorn temperature is higher than the traditional one.

The above comments apply to heterotic strings. However, as we shall see, similar remarks will also apply for certain Type II strings in dimensions  $D < 10$ .

In Sect. 6, we shall then broaden our discussion to investigate the appearance of other Hagedorn-like transitions. We shall find that many such additional transitions may exist, but that they are generally quite model-dependent.

Much of the discussion in this paper will be as general and model-independent as possible. Consequently, our focus will primarily be on the Hagedorn transition, and not on the model-dependent issue of determining the correct finite-temperature extrapolation of zero-temperature string models. To compensate for this, in Appendix A we will explicitly analyze the case of the ten-dimensional supersymmetric  $SO(32)$  heterotic string. We shall explicitly construct what we believe is the finite-temperature extrapolation for this theory, and in the process lay out our general criteria for such extrapolations. Moreover, using these criteria, we shall show that each of the tachyon-free ten-dimensional heterotic strings experiences a re-identified Hagedorn transition at a temperature normally associated with Type II strings. This includes not only the supersymmetric  $SO(32)$  and  $E_8 \times E_8$  heterotic strings, but also the ten-dimensional tachyon-free non-supersymmetric  $SO(16) \times SO(16)$  heterotic string [24, 25]. Thus, we conclude that *all tachyon-free ten-dimensional closed strings actually have a universal Hagedorn temperature*. We shall also comment on the possibility of thermal spin-statistics violations at the Planck scale, and show that this possibility is intimately connected with the re-identification of the Hagedorn phenomenon.

## 2 Calculating thermodynamic potentials in string theory

We begin by quickly reviewing the calculation of one-loop thermodynamic quantities in closed string theories. Our purpose here is merely to recall established formalism and set notational conventions. For detailed derivations or explanations, we refer the reader to the original literature or Ref. [4].

Just as in ordinary statistical mechanics, the fundamental quantity of interest in string thermodynamics is the one-loop thermal string partition function  $Z_{\text{string}}(\tau, T)$ . This is a function of not only the temperature  $T$  but also the complex modular parameter  $\tau$  (parametrizing the shape of the one-loop toroidal string worldsheet). In terms of  $Z_{\text{string}}(\tau, T)$ , the one-loop thermal vacuum amplitude is then given by the one-loop modular integral

$$\mathcal{V}(T) \equiv -\frac{1}{2} \mathcal{M}^{D-1} \int_{\mathcal{F}} \frac{d^2\tau}{(\text{Im } \tau)^2} Z_{\text{string}}(\tau, T) \quad (2.1)$$

where  $\mathcal{M} \equiv M_{\text{string}}/(2\pi)$  is the reduced string scale;  $D$  is the number of non-compact spacetime dimensions; and where

$$\mathcal{F} \equiv \{\tau : |\text{Re } \tau| \leq \frac{1}{2}, \text{Im } \tau > 0, |\tau| \geq 1\} \quad (2.2)$$

is the fundamental domain of the modular group. For future notational convenience, we shall define  $\tau_1 \equiv \text{Re } \tau$  and  $\tau_2 \equiv \text{Im } \tau$ . In general,  $\mathcal{V}(T)$  plays the role usually taken by the logarithm of the statistical-mechanical partition function. Given this definition for  $\mathcal{V}$ , the free energy  $F$ , internal energy  $U$ , entropy  $S$ , and specific heat  $c_V$  then follow from the standard thermodynamic definitions:

$$F = T\mathcal{V}, \quad U = -T^2 \frac{d}{dT}\mathcal{V}, \quad S = -\frac{d}{dT}F, \quad c_V = \frac{d}{dT}U. \quad (2.3)$$

Because of its central role in determining the thermodynamics of the corresponding string theory, we shall now focus on the calculation of the string thermal partition function  $Z_{\text{string}}(\tau, T)$ . Let us begin by discussing the case of a compactified bosonic string at zero temperature. In such a case, we have

$$Z_{\text{model}}(\tau) \equiv \text{Tr } \bar{q}^{H_R} q^{H_L} \quad (2.4)$$

where the trace is over the complete Fock space of states in the theory. Here  $q \equiv \exp(2\pi i\tau)$ , and  $(H_R, H_L)$  denote the worldsheet energies for the right- and left-moving worldsheet degrees of freedom, respectively. For example, in the case of the bosonic string compactified to  $D$  spacetime dimensions,  $Z_{\text{model}}$  takes the general form

$$Z_{\text{model}}(\tau) = \tau_2^{1-D/2} \frac{\overline{\Theta}^{26-D} \Theta^{26-D}}{\bar{\eta}^{24} \eta^{24}} \quad (2.5)$$

where the numerator  $\overline{\Theta}^{26-D} \Theta^{26-D}$  schematically represents a sum over the  $2(26-D)$ -dimensional compactification lattice for left- and right-movers. Note that in general,  $Z_{\text{model}}$  is the quantity which appears in the calculation of the one-loop cosmological constant (vacuum energy density) of the model:

$$\Lambda \equiv -\frac{1}{2} \mathcal{M}^D \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} Z_{\text{model}} \quad (2.6)$$

Of course, this quantity is divergent for the compactified bosonic string as a result of the physical bosonic-string tachyon.

In order to extend such a model to finite temperature, we recall that in string theory (just as in ordinary quantum field theory), finite-temperature effects can be incorporated [4, 26] by compactifying an extra (Euclidean) time dimension on a circle of radius  $R_T = (2\pi T)^{-1}$ . The Matsubara modes are nothing but the Kaluza-Klein states corresponding to this compactification. Since all of the states in these string models are presumed to be bosonic, and since each bosonic state must be assigned periodic boundary conditions around this extra dimension, we see that the Matsubara frequencies for each of the zero-temperature string states have integer modings. However, for extended objects such as closed strings, we must include not only “momentum” Matsubara states (as described above), but also “winding” Matsubara states (analogues of the usual string winding modes). Both types of states are necessary for the modular invariance of the underlying theory at finite temperature. We thus obtain a full, thermal partition function of the form

$$Z_{\text{string}}(\tau, T) \equiv Z_{\text{model}}(\tau) Z_{\text{circ}}(\tau, T) \quad (2.7)$$

where the extra factor  $Z_{\text{circ}}$  represents a double summation over integer Matsubara momentum and winding modes:

$$Z_{\text{circ}}(\tau, T) = \sqrt{\tau_2} \sum_{m,n \in \mathbb{Z}} \bar{q}^{(ma-n/a)^2/4} q^{(ma+n/a)^2/4} \quad (2.8)$$

with  $a \equiv 2\pi T/M_{\text{string}} \equiv T/\mathcal{M}$ . It is the full thermal partition function  $Z_{\text{string}}(\tau, T)$  which is then used in the calculation of the vacuum amplitude in Eq. (2.1). Note that  $Z_{\text{circ}} \rightarrow 1/a$  as  $a \equiv T/\mathcal{M} \rightarrow 0$ . We therefore find that  $\mathcal{V}(T) \rightarrow \Lambda/T$ , and thus  $F(T) \rightarrow \Lambda$ , as  $T/\mathcal{M} \rightarrow 0$ . By contrast, we have  $\mathcal{V}(T) \rightarrow \Lambda T/\mathcal{M}^2$  as  $T/\mathcal{M} \rightarrow \infty$ , implying that  $F(T) \rightarrow \Lambda T^2/\mathcal{M}^2$ .

Let us now proceed to discuss the more general case of fermionic string theories (such as the superstring and the heterotic string). The critical differences relative to the bosonic string are the presence of spacetime fermions in the spacetime spectrum and the possibility of removing on-shell tachyons through non-trivial GSO projections.

Once again, let us begin by considering the zero-temperature theory. The partition function for such a theory takes the form

$$Z_{\text{model}}(\tau) \equiv \text{Tr} (-1)^F \bar{q}^{H_R} q^{H_L}, \quad (2.9)$$

which is completely analogous to Eq. (2.4) except for the spacetime statistics factor  $(-1)^F$ , where  $F$  is the spacetime fermion number. Thus bosonic states contribute positively to  $Z_{\text{model}}$ , while fermionic states contribute negatively. Given this, the one-loop cosmological constant is given by the same one-loop integral in Eq. (2.6); however, the presence of non-trivial GSO projections can now eliminate on-shell tachyons and result in a finite cosmological constant.

In order to extend this theory to finite temperature, we again must introduce the summations over thermal Matsubara states. However, it is here that the primary difference arises: while bosonic states must be periodic around the extra (Euclidean) time direction, resulting in integer-moded Matsubara frequencies, the fermionic states must be *antiperiodic* around this direction, resulting in Matsubara modings which are *integer plus one-half*. Thus, the bosonic and fermionic portions of  $Z_{\text{model}}$  must be multiplied by different Matsubara sums, destroying the simple factorized form in Eq. (2.7). In general, this structure can also be further complicated by subsequent orbifolding and the introduction of temperature-dependent Wilson lines. In such cases, modular invariance can serve as a useful tool for constraining the form of the resulting partition functions [27, 28, 8, 10], but other physical criteria (such as proper thermal spin-statistics) play an important role.

Towards this end, let us introduce [27] four new functions  $\mathcal{E}_{0,1/2}$  and  $\mathcal{O}_{0,1/2}$  which are the same as the summation in  $Z_{\text{circ}}$  in Eq. (2.8) except for the following restrictions on their summation variables:<sup>\*</sup>

$$\begin{aligned}\mathcal{E}_0 &= \{m \in \mathbb{Z}, n \text{ even}\} \\ \mathcal{E}_{1/2} &= \{m \in \mathbb{Z} + \frac{1}{2}, n \text{ even}\} \\ \mathcal{O}_0 &= \{m \in \mathbb{Z}, n \text{ odd}\} \\ \mathcal{O}_{1/2} &= \{m \in \mathbb{Z} + \frac{1}{2}, n \text{ odd}\}.\end{aligned}\tag{2.10}$$

Under the modular transformation  $T : \tau \rightarrow \tau + 1$ , the first three functions are invariant while  $\mathcal{O}_{1/2}$  picks up a minus sign; likewise, under  $S : \tau \rightarrow -1/\tau$ , these functions mix according to

$$\begin{pmatrix} \mathcal{E}_0 \\ \mathcal{E}_{1/2} \\ \mathcal{O}_0 \\ \mathcal{O}_{1/2} \end{pmatrix}(-1/\tau) = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} \mathcal{E}_0 \\ \mathcal{E}_{1/2} \\ \mathcal{O}_0 \\ \mathcal{O}_{1/2} \end{pmatrix}(\tau).\tag{2.11}$$

Note that in the  $T/\mathcal{M} \rightarrow 0$  limit,  $\mathcal{O}_0$  and  $\mathcal{O}_{1/2}$  each vanish while  $\mathcal{E}_0, \mathcal{E}_{1/2} \rightarrow \mathcal{M}/T$ ; by contrast, as  $T/\mathcal{M} \rightarrow \infty$ ,  $\mathcal{E}_{1/2}$  and  $\mathcal{O}_{1/2}$  each vanish while  $\mathcal{E}_0, \mathcal{O}_0 \rightarrow T/(2\mathcal{M})$ . Clearly,  $\mathcal{E}_0 + \mathcal{O}_0 = Z_{\text{circ}}$ .

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<sup>\*</sup>Note that these functions are to be distinguished from a related (and also often used) set of functions with the same names in which the roles of  $m$  and  $n$  are exchanged.

In terms of these functions, our complete thermal string partition function for fermionic string theories then generically takes the form [27, 28, 8, 10]

$$\begin{aligned} Z_{\text{string}}(\tau, T) &= Z^{(1)}(\tau) \mathcal{E}_0(\tau, T) + Z^{(2)}(\tau) \mathcal{E}_{1/2}(\tau, T) \\ &\quad + Z^{(3)}(\tau) \mathcal{O}_0(\tau, T) + Z^{(4)}(\tau) \mathcal{O}_{1/2}(\tau, T). \end{aligned} \quad (2.12)$$

Clearly, the individual blocks  $Z^{(i)}$  must transform under modular transformations in such a way that  $Z_{\text{string}}$  is modular invariant; in this case, this implies that each  $Z^{(i)}$  must transform exactly as does its corresponding  $\mathcal{E}/\mathcal{O}$  function. Note that the original corresponding zero-temperature partition function is given by

$$Z_{\text{model}} = Z^{(1)} + Z^{(2)}, \quad (2.13)$$

since these are the only two terms which survive the  $T/\mathcal{M} \rightarrow 0$  limit. We thus continue to find the asymptotic behavior  $\mathcal{V}(T) \rightarrow \Lambda/T$  as  $T/\mathcal{M} \rightarrow 0$ .

It is interesting to note that the opposite limit as  $T/\mathcal{M} \rightarrow \infty$  results in the behavior  $V(T) \rightarrow \tilde{\Lambda}T/(2\mathcal{M}^2)$  and  $F(T) \rightarrow \tilde{\Lambda}T^2/(2\mathcal{M}^2)$  where  $\tilde{\Lambda}$  is the cosmological constant associated with the alternate zero-temperature model whose partition function is given by

$$\tilde{Z}_{\text{model}} = Z^{(1)} + Z^{(3)}. \quad (2.14)$$

Thus, the thermal partition function (2.12) can be viewed as mathematically *interpolating* between one zero-temperature string model at  $T = 0$  and a *different* zero-temperature string model as  $T \rightarrow \infty$ . The extra factor of two in the asymptotic behavior in the infinite-temperature limit arises from the reduction of the volume of the  $\mathbb{Z}_2$  orbifold that implements the appropriate thermal twists for the spacetime fermions.

Note that the  $\mathcal{E}/\mathcal{O}$  functions satisfy the identities

$$\begin{aligned} \mathcal{E}_0(1/a) &= \mathcal{E}_0(2a), & \mathcal{E}_{1/2}(1/a) &= \mathcal{O}_0(2a) \\ \mathcal{O}_0(1/a) &= \mathcal{E}_{1/2}(2a), & \mathcal{O}_{1/2}(1/a) &= \mathcal{O}_{1/2}(2a) \end{aligned} \quad (2.15)$$

where  $a = T/\mathcal{M}$ . Thus, for every partition function of the form in Eq. (2.12), there is another in which we replace  $a \rightarrow 2/a$  and exchange  $Z^{(2)}$  and  $Z^{(3)}$ . This has the net effect of preserving the interpolation, but exchanging the  $T \rightarrow 0$  and  $T \rightarrow \infty$  limits.

We see, then, that a zero-temperature model whose partition function is given in Eq. (2.13) will have a finite-temperature extrapolation of the form in Eq. (2.12). It is important to note, however, that the particular form of Eq. (2.12) is not uniquely determined simply by the zero-temperature partition function sum  $Z_{\text{model}}$  in Eq. (2.13); it also depends on how  $Z_{\text{model}}$  is divided into  $Z^{(1)}$  and  $Z^{(2)}$ . In other words, modular invariance alone is not sufficient to determine the unique finite-temperature extrapolation of a given zero-temperature model unless one assumes that all bosonic states in  $Z_{\text{model}}$  are part of  $Z^{(1)}$  and all fermionic states are part of  $Z^{(2)}$ . As mentioned above,

this simple assumption may be affected by orbifold twists and string consistency constraints. We shall discuss this issue further in Appendix A.

Finally, it is also easy to see from Eq. (2.12) why thermal effects break space-time supersymmetry. In the  $T \rightarrow 0$  limit, our partition function reduces to  $Z_{\text{model}} = Z^{(1)} + Z^{(2)}$ ; if this limit is supersymmetric, then we necessarily have  $Z^{(1)} = -Z^{(2)}$  as an identity on the  $q$ -expansions of these expressions, resulting in the zero-temperature supersymmetric limit  $Z_{\text{model}} = 0$ , with  $\Lambda = 0$ . However, even though this partition function vanishes at zero temperature, this cancellation will not persist at non-zero temperatures because  $Z^{(1)}$  and  $Z^{(2)}$  are multiplied by different  $\mathcal{E}/\mathcal{O}$  functions representing the traces over different thermal Matsubara and/or winding modes. This is the reflection of the fact that thermal effects necessarily treat bosons and fermions differently, thereby breaking any supersymmetry which may have existed at zero temperature.

### 3 The usual Hagedorn transition: Standard arguments

Our main concern in this paper is to demonstrate that the traditional Hagedorn phenomenon is non-existent for a wide class of closed strings (including all heterotic strings), and should actually be re-identified. Let us therefore first review the standard arguments for the appearance of the Hagedorn transition. In Sect. 4, we shall then discuss why this transition is, in fact, absent.

#### 3.1 The UV argument

The usual ultraviolet (UV) argument for the existence of a Hagedorn temperature involves the exponential rise in the number of string states as a function of mass. In a nutshell, if  $g_M$  denotes the number of states with mass  $M$ , then the thermal partition function is given by  $Z(T) = \sum g_M e^{-M/T}$ . However, if  $g_M \sim e^{\alpha M}$  as  $M \rightarrow \infty$ , then  $Z(T)$  diverges for  $T \geq 1/\alpha$ . This suggests the existence of some sort of phase transition at the critical (Hagedorn) temperature  $T_H \equiv 1/\alpha$ .

Let us now develop this argument in the precise language of our string partition functions. We begin by considering the zero-temperature (*i.e.*, non-thermal) partition function  $Z_{\text{model}}$ . Since this partition function represents a trace over the string Fock space as in Eq. (2.9), it encodes the information about the net degeneracies of string states at each mass level in the zero-temperature theory. Specifically, if we expand  $Z_{\text{model}}$  as a power series in  $q$  and  $\bar{q}$ , we obtain an expression of the form\*

$$Z_{\text{model}}(\tau) = \tau_2^{1-D/2} \sum_{\tilde{m}, \tilde{n}} a_{\tilde{m}\tilde{n}} \bar{q}^{\tilde{m}} q^{\tilde{n}} . \quad (3.1)$$

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\*We designate our summation variables as  $\tilde{m}$  and  $\tilde{n}$  in order to distinguish them from the thermal momentum and winding momentum modes  $m$  and  $n$  in Eqs. (2.8) and (2.10).

Here  $(\tilde{m}, \tilde{n})$  represent the possible eigenvalues of the right- and left-moving worldsheet Hamiltonians  $(H_R, H_L)$ , and  $a_{\tilde{m}\tilde{n}}$  represents the number of bosonic minus fermionic states which actually have those eigenvalues and satisfy the GSO constraints. Modular invariance requires that  $\tilde{m} - \tilde{n} \in \mathbb{Z}$  for all  $a_{\tilde{m}\tilde{n}} \neq 0$ ; a state is said to be “on-shell” or “level-matched” if  $\tilde{m} = \tilde{n}$ . Note that the  $\tau_2^{1-D/2}$  prefactor represents the result of the integration (*i.e.*, the trace) over the continuous spectrum of states corresponding to the uncompactified dimensions.

In general, the spacetime mass of an arbitrary  $(\tilde{m}, \tilde{n})$  state is given by

$$\alpha' M_{\tilde{m}\tilde{n}}^2 = 2(\tilde{m} + \tilde{n}) \quad (3.2)$$

where  $\alpha' = 1/M_{\text{string}}^2$ . Thus, we see that states for which  $\tilde{m} + \tilde{n} \geq 0$  are massive and/or massless, while states with  $\tilde{m} + \tilde{n} < 0$  are tachyonic. In general, one constructs a consistent string model in such a way as to avoid on-shell tachyons (*i.e.*, to ensure that  $a_{\tilde{m}\tilde{n}} = 0$  for all  $\tilde{m} = \tilde{n} < 0$ ). Of course, if the model exhibits spacetime supersymmetry at zero temperature, then  $a_{\tilde{m}\tilde{n}} = 0$  for all  $\tilde{m}, \tilde{n}$ , but the supersymmetry will be broken by finite-temperature effects which eventually multiply the bosonic and fermionic contributions to  $Z_{\text{model}}$  with different thermal  $\mathcal{E}/\mathcal{O}$  functions. Therefore, for the purposes of the traditional Hagedorn derivation, we shall consider the bosonic and fermionic contributions to  $a_{\tilde{m}\tilde{n}}$  separately.

In order to discuss the behavior of  $a_{\tilde{m}\tilde{n}}$ , let us first recall that  $Z_{\text{model}}$  can generally be constructed in terms of the characters  $\chi_i$  and  $\bar{\chi}_{\bar{i}}$  of the corresponding left- and right-moving conformal field theories (CFTs) on the string worldsheet:

$$Z_{\text{model}}(\tau) = \tau_2^{1-D/2} \sum_{\bar{i}i} \bar{\chi}_{\bar{i}}(\bar{\tau}) N_{\bar{i}i} \chi_i(\tau) . \quad (3.3)$$

The coefficients  $N_{\bar{i}i}$  describe the manner in which the left- and right-moving CFTs are stitched together, and encode the information concerning the GSO projections inherent in the given string model. In general, each character  $\chi_i$  represents a trace over the sector of the CFT corresponding to a specific primary field  $\phi_i$ , and has a  $q$ -expansion of the form

$$\chi_i(\tau) = q^{h_i - c/24} \sum_{p=0}^{\infty} a_p^{(i)} q^p , \quad (3.4)$$

where  $c$  is the central charge of the CFT and  $h_i$  is the conformal weight of the primary field  $\phi_i$ . Moreover, the coefficients  $a_p^{(i)}$  count the number of descendent fields, and are known to grow asymptotically as [29, 30, 31]

$$a_p^{(i)} \sim \exp\left(4\pi\sqrt{\frac{cp}{24}}\right) \quad \text{as } p \rightarrow \infty . \quad (3.5)$$

Note that this rate of asymptotic growth applies for all sectors of the CFT.

Thus, combining our results for the left- and right-movers, we see that the numbers of on-shell bosonic or fermionic states in a given string theory generally grow as

$$a_{\tilde{n}\tilde{n}} \sim \exp \left[ 4\pi \left( \sqrt{\frac{c_L}{24}} + \sqrt{\frac{c_R}{24}} \right) \sqrt{\tilde{n}} \right] \quad \text{as } \tilde{n} \rightarrow \infty , \quad (3.6)$$

where  $c_{L,R}$  are the central charges of the left- and right-moving worldsheet CFTs in light-cone gauge. Note that in this result, we have set  $\tilde{m} = \tilde{n}$  because we are focusing on the on-shell states exclusively.

Eq. (3.6) describes the rate of exponential growth in the numbers of physical bosonic and/or fermionic string states in terms of the central charges  $c_{L,R}$  of the left- and right-moving worldsheet CFTs in light-cone gauge. As such, this result is completely general, and applies to all bosonic strings, superstrings, or heterotic strings.

Using this result, the standard argument for the Hagedorn transition proceeds by taking this collection of states and calculating the thermal partition function as we would in ordinary point-particle theories. Specifically, we substitute the degeneracies  $a_{\tilde{n}\tilde{n}}$  from Eq. (3.6) into a general partition function of the form

$$Z = \sum_{\tilde{n}} a_{\tilde{n}\tilde{n}} e^{-M_{\tilde{n}\tilde{n}}/T} \quad (3.7)$$

where  $T$  is the temperature and where  $M_{\tilde{n}\tilde{n}} = 2\sqrt{\tilde{n}}M_{\text{string}}$  is the spacetime mass corresponding to the  $\tilde{m} = \tilde{n}$  level, as given in Eq. (3.2). We thus immediately find that this partition function will diverge for temperatures  $T \geq T_H$ , where

$$T_H = \frac{1}{2\pi} \left( \sqrt{\frac{c_L}{24}} + \sqrt{\frac{c_R}{24}} \right)^{-1} M_{\text{string}} . \quad (3.8)$$

This is therefore identified as the Hagedorn temperature. For the bosonic string, we have the light-cone gauge central charges  $(c_L, c_R) = (24, 24)$ , while for superstrings (Type II strings) we have  $(c_L, c_R) = (12, 12)$  and for heterotic strings we have  $(c_L, c_R) = (24, 12)$ . We thus find that

$$\frac{T_H}{M_{\text{string}}} = \begin{cases} (4\pi)^{-1} & \text{for the bosonic string,} \\ (2\sqrt{2}\pi)^{-1} & \text{for the Type II superstring,} \\ [(2 + \sqrt{2})\pi]^{-1} & \text{for the heterotic string.} \end{cases} \quad (3.9)$$

These are indeed the traditional Hagedorn temperatures normally associated with these theories.

### 3.2 The IR argument

The alternative, infrared (IR) way of understanding the emergence of the Hagedorn transition involves the *low-lying* string states. As we have seen, the lowest-lying

string state is, *a priori*, the tachyonic string ground state. This in turn arises from the identity sectors of the respective right- and left-moving worldsheet CFTs, with  $h_{\bar{i}} = h_i = 0$ . Thus, according to Eq. (3.4), the string ground state has right- and left-moving worldsheet energies

$$(H_R, H_L) = \left( -\frac{c_R}{24}, -\frac{c_L}{24} \right). \quad (3.10)$$

While this state is necessarily tachyonic at zero temperature, at non-zero temperatures it gives rise to an infinite tower of associated thermal momentum and winding modes. As a result of additional thermal mass contributions that result from such momenta and windings, such thermal states can be massless or even massive. The magnitudes of these mass contributions are generally encoded within the  $\mathcal{E}/\mathcal{O}$  functions which, according to Eq. (2.8), correspond to the additional worldsheet energies

$$(\Delta H_R, \Delta H_L) = \left[ \frac{1}{4}(ma - n/a)^2, \frac{1}{4}(ma + n/a)^2 \right] \quad (3.11)$$

where  $a \equiv T/\mathcal{M}$  and where  $(m, n)$  are the thermal momentum and winding numbers. Thus, in order for one of the thermal excitation modes of the string ground state to become massless, we must satisfy the constraints

$$\begin{aligned} ma + n/a &= \pm 2\sqrt{c_L/24} \\ ma - n/a &= \pm 2\sqrt{c_R/24} \end{aligned} \quad (3.12)$$

where the two  $\pm$  signs are uncorrelated. Taking the difference of these two constraint equations as well as the difference of their squares then yields:

$$n/a = \pm \left( \sqrt{\frac{c_L}{24}} \pm \sqrt{\frac{c_R}{24}} \right), \quad mn = \frac{c_L - c_R}{24}, \quad (3.13)$$

where again the signs in the first equation are uncorrelated.

Since the  $(m, n)$  quantum numbers are also subject to the restrictions appropriate for the particular  $\mathcal{E}/\mathcal{O}$  functions listed in Eq. (2.10), we see that the second constraint in Eq. (3.13) has only two solutions with non-zero thermal winding number for each string:

$$\begin{aligned} \text{bosonic, Type II :} \quad m &= 0, \quad n = \pm 1, \\ \text{heterotic :} \quad m &= \pm \frac{1}{2}, \quad n = \pm 1, \end{aligned} \quad (3.14)$$

where the signs in the second line are now correlated. Thus, inserting  $n = \pm 1$  into the first constraint in Eq. (3.13), we find that there are generally only two positive solutions for  $a$ :

$$a^{(\pm)} = \left| \sqrt{\frac{c_L}{24}} \pm \sqrt{\frac{c_R}{24}} \right|^{-1}. \quad (3.15)$$

These correspond to the temperatures

$$T^{(\pm)} = \frac{1}{2\pi} \left| \sqrt{\frac{c_L}{24}} \pm \sqrt{\frac{c_R}{24}} \right|^{-1} M_{\text{string}} . \quad (3.16)$$

Note that  $T^{(+)} < T^{(-)}$ .

It is easy to interpret these equations physically. At small temperatures, the thermal excitation of the string ground state in Eq. (3.14) is extremely massive. This is a direct result of the non-trivial thermal winding number in Eq. (3.14), which provides a huge mass contribution at small temperatures. However, as the temperature increases, this additional mass contribution becomes smaller and smaller until eventually this state becomes massless. This occurs at the temperature  $T^{(+)}$ , which is of course in complete agreement with the Hagedorn temperature obtained in Eq. (3.8). Thus, we see that the transition triggered by the appearance of the new massless state at  $T^{(+)}$  is nothing but the Hagedorn transition. Indeed, formally continuing beyond  $T^{(+)}$ , we would find that this state becomes tachyonic, in agreement with our expectations of a phase transition.

It is also possible to interpret  $T^{(-)}$  in the case of the heterotic string. If we were to continue to formally increase the temperature beyond  $T^{(+)}$ , this tachyonic state would ultimately reach a maximum depth before turning around and becoming more massive again as a result of the non-zero heterotic thermal *momentum* mode with  $m = \pm 1/2$ . This state would ultimately become massless again at the higher temperature  $T^{(-)}$ . In the case of the bosonic or Type II strings, by contrast, our Hagedorn state has a vanishing thermal winding number. It therefore becomes increasingly tachyonic as we increase the temperature beyond  $T^{(+)}$ , ultimately settling into the original tachyonic ground state as  $T \rightarrow \infty$ . Thus, in such cases, we see that  $T^{(-)}$  is essentially infinite. Of course, these are only formal interpretations of the above equations since we should not extrapolate our calculation beyond the Hagedorn transition at  $T^{(+)}$ .

Note that in all cases, the Hagedorn temperature  $T^{(+)}$  calculated through this tachyonic winding-mode argument agrees with the Hagedorn temperature calculated through the exponential density of states argument in Eq. (3.8). As we can see from the above derivations, this agreement arises because the rate of exponential growth of the density of states given in Eq. (3.6) is correlated with the depth of the original tachyonic string ground state given in Eq. (3.10), with both quantities set by the central charge of the string worldsheet theory.

## 4 Why the standard Hagedorn arguments fail

So what fails in the above arguments? In this section, we shall describe why the above Hagedorn transition is, in fact, completely invisible in calculations of the standard thermodynamic quantities. We shall concentrate on the case of the heterotic string, deferring our discussion of Type II strings until the end of this section.

## 4.1 The UV argument

Let us begin by discussing the ultraviolet derivation based on the exponentially rising density of states. As we have seen in Eq. (3.6), the numbers of on-shell bosonic and fermionic states in a given string theory generically grow exponentially with the spacetime mass. In Eq. (3.7), these degeneracies are inserted into a point-particle partition function, leading to a divergence above the Hagedorn temperature. The problem, of course, is that the partition function in Eq. (3.7) is not a proper string-theoretic partition function; it assumes that the string is nothing but a collection of the states to which its excitations give rise. Instead, however, we must do a proper string-theoretic vacuum-amplitude calculation as outlined in Eq. (2.1), using a string partition function which depends not only on the temperature  $T$  but also a torus parameter  $\tau$ .

There are three major differences between the point-particle partition function in Eq. (3.7) and the proper string partition function. The first is that in point-particle field theories, we have only thermal momentum (Matsubara) modes; there are no analogues of string thermal winding modes. However, while very important in other contexts (*e.g.*, in studies of thermal duality symmetries [7, 28, 8, 32, 33]), this difference plays no essential role here. The second difference is the temperature dependence: Eq. (3.7) employs a standard Boltzman suppression factor, while the thermal components in Eqs. (2.8) and (2.10) are quadratic in temperature exponential. Once again, however, this difference is not relevant in resolving the issue of the missing Hagedorn transition. Indeed, upon performing the  $\tau$ -integration, the quadratic temperature dependence in Eqs. (2.8) and (2.10) gives rise to the standard Boltzman suppression factor in the high-temperature (or high-energy) limits.

The third difference, however, is more significant and goes right to the heart of string theory. Using various Schwinger proper-time identities, it is always possible to recast our field-theoretic partition function into a form resembling Eq. (2.1); in such a form, the quantity  $\tau$  emerges as the Schwinger proper time. However, in this form we would then be instructed to integrate  $\tau$  over the strip  $\mathcal{S}$  defined as

$$\mathcal{S} \equiv \{\tau : |\tau_1| \leq 1/2, \tau_2 \geq 0\} . \quad (4.1)$$

This is to be contrasted with the string calculation, where modular invariance requires us to restrict our  $\tau$ -integration to the fundamental domain  $\mathcal{F}$  defined in Eq. (2.2). This difference has a major effect because the  $\mathcal{F}$  domain avoids the ultraviolet  $\tau_2 \rightarrow 0$  region completely, and this is the region of integration which ultimately gives rise to the Hagedorn divergence.

To see this, note that if we expand each  $Z^{(i)}$  in Eq. (2.12) in the form of Eq. (3.1), we can combine Eqs. (3.2) and (2.8) to write  $Z_{\text{string}}$  in the form

$$Z_{\text{string}}(\tau, T) = \tau_2^{(3-D)/2} \sum_{\tilde{m}, \tilde{n}} \sum_{m, n} a_{\tilde{m}\tilde{n}} e^{2\pi i \tau_1 (\tilde{n} - \tilde{m} + mn)} e^{-\pi \tau_2 \alpha' M_{\tilde{m}\tilde{n}}^2} e^{-\pi \tau_2 \alpha' M_{mn}^2} \quad (4.2)$$

where we have separated the “bare” zero-temperature mass  $M_{\tilde{m}\tilde{n}}$  in Eq. (3.2) from the additional “thermal” mass  $M_{mn}$ :

$$\alpha' M_{mn}^2 \equiv \frac{m^2 T^2}{\mathcal{M}^2} + \frac{n^2 \mathcal{M}^2}{T^2} . \quad (4.3)$$

Of course, the  $\tau_1$  exponential in Eq. (4.2) is generally trivial: demanding that  $Z_{\text{string}}$  be invariant under the modular transformation  $\tau \rightarrow \tau + 1$  yields the constraint

$$\tilde{n} - \tilde{m} + mn \in \mathbb{Z} , \quad (4.4)$$

and integrating this exponential over  $\tau_1$  across the region  $-1/2 \leq \tau_1 \leq 1/2$  enforces the stronger level-matching constraint  $\tilde{n} - \tilde{m} + mn = 0$  for on-shell string states. However, our interest is in the behavior of the degeneracies  $a_{\tilde{n}\tilde{n}}$ . Recall from Eqs. (3.6) and (3.8) that

$$a_{\tilde{n}\tilde{n}} \sim e^{M_{\tilde{n}\tilde{n}}/T_H} \quad \text{as } \tilde{n} \rightarrow \infty , \quad (4.5)$$

where we are restricting our attention to  $\tilde{m} = \tilde{n}$  (as would be appropriate, for example, in the  $m = n = 0$  thermal sector); this is the exponential growth that generically leads to a Hagedorn transition. However, inserting Eq. (4.5) into Eq. (4.2) and performing the sum over  $\tilde{n}$ , we see that the Hagedorn growth in Eq. (4.5) is suppressed by the additional  $\tau_2$  exponential in Eq. (4.2) for all  $\tau_2 > 0$ . Indeed, the string partition function diverges only in the  $\tau_2 \rightarrow 0$  region. Thus, the truncation of the region of integration from  $\mathcal{S}$  to  $\mathcal{F}$  in string theory is ultimately responsible for regulating the ultraviolet behavior and thereby completely eliminating the Hagedorn transition. Indeed, this observation has already been made in the recent string literature [33].

Note that in this argument, we have been assuming that there are no physical tachyonic states contributing to Eq. (4.2). Indeed, the purpose of this argument has merely been to demonstrate that the UV asymptotic rise in the degeneracy of states does not, in and of itself, lead to a divergence in the closed-string thermal amplitude.

It is, of course, always possible as a mathematical exercise to rewrite our integral over  $\mathcal{F}$  as an integral over  $\mathcal{S}$  using an infinite set of modular transformations and Poisson resummations [7, 34]. However, this is merely a mathematical rewriting, and thus cannot introduce a divergence where none exists. Indeed, even in such a strip-based representation for the vacuum amplitudes, there will be delicate algebraic cancellations which eliminate the supposed UV divergence as  $\tau_2 \rightarrow 0$ . In the case of the heterotic string which is our focus in this section, these cancellations arise through additional phases in the strip representation which emerge as the result of the thermal  $\mathbb{Z}_2$  orbifolding for worldsheet fermions.

## 4.2 The IR argument

It is also important to understand the elimination of the Hagedorn transition from the perspective of the infrared argument involving low-energy thermal winding

state which becomes massless at the Hagedorn temperature. However, once again, it is easy to see where the error in this argument lies. Recall that this argument makes use of thermal excitations of the string ground state in Eq. (3.10). However, this assumes that this ground state is actually present in the string spectrum — *i.e.*, that it satisfies the appropriate finite-temperature GSO constraints and appears in the actual finite-temperature string partition function. In other words, it is assumed that  $Z_{\text{string}}$  [or, more precisely, one of the  $Z^{(i)}$  functions in Eq. (2.12)] actually contains a term of the form

$$\bar{q}^{-c_R/24} q^{-c_L/24}, \quad (4.6)$$

corresponding to the tachyonic string ground state. Equivalently, phrased in terms of CFT characters of the form in Eq. (3.3), it is assumed that  $Z_{\text{string}}$  contains a term of the form

$$\bar{\chi}_I \chi_I, \quad (4.7)$$

representing a tensor product of the identity sectors of the right- and left-moving worldsheet CFTs. Unfortunately, this is generally not the case: in the heterotic string, this state is always GSO-projected out of the spectrum. This is certainly true in the zero-temperature theory. However, our claim is that this is also true in the *finite-temperature* theory, even with its modified GSO constraints, once the correct extrapolation to finite temperature is identified. Thus, there is no state which survives the GSO projection in order to trigger the Hagedorn transition.

It is easy to see heuristically why this state must be projected out in the case of the heterotic string. Since the ground state of the heterotic string has worldsheet energies  $(\tilde{m}, \tilde{n}) = (-1/2, -1)$ , we see from the modular-invariance constraint in Eq. (4.4) that such a state could only appear in the term  $Z^{(4)}$ , which multiplies the thermal function  $\mathcal{O}_{1/2}$  in Eq. (2.12). However, this represents a twisted sector of the  $\mathbb{Z}_2$  thermal orbifold, and we do not expect to see the ground state of a conformal field theory emerging from a twisted sector. Equivalently stated, we expect a term of the form in Eq. (4.7) to appear not within  $Z^{(4)}$ , but rather from the untwisted sectors corresponding to  $Z^{(1)}$ ,  $Z^{(2)}$ , or even  $Z^{(3)}$ . However, modular invariance (specifically invariance under  $\tau \rightarrow \tau + 1$ ) prevents this from happening. Thus, we conclude that this state must be GSO-projected out of the spectrum in any self-consistent thermal extrapolation of a zero-temperature heterotic string model.

This is clearly an important point, and we stress that it requires a proper understanding of the manner in which a zero-temperature string model can be self-consistently extrapolated to non-zero temperature. While it is certainly consistent *at the level of modular invariance* for the CFT ground state  $\bar{\chi}_I \chi_I$  to appear within  $Z^{(4)}$ , our claim is that this is not consistent with a proper worldsheet construction for a string theory compactified on a thermal circle, along with an appropriate  $\mathbb{Z}_2$  orbifold for fermionic statistics. Our claim, then, is that any proper finite-temperature extrapolation will have implicit finite-temperature GSO constraints which eliminate this state from  $Z^{(4)}$ . We shall discuss this conclusion further in Appendix A, along

with explicit examples.

Note, in this context, that the two reasons for the absence of the Hagedorn transition in the heterotic string are actually correlated with each other: the GSO projections which remove the ground state from the finite-temperature string spectrum are the same projections which ensure the modular invariance that allows us to truncate the region of integration from the strip  $\mathcal{S}$  to the fundamental domain  $\mathcal{F}$  and avoid on-shell tachyons. Thus, once again, our ultraviolet and infrared Hagedorn arguments are ultimately correlated. Essentially, the usual arguments for the Hagedorn transition assume that the string is nothing more than a tensor product of worldsheet conformal field theories (CFTs), with a tensor-product ground state and a tensor-product degeneracy of states. However, the GSO projections operate *across* the tensor product of CFT's, deforming this structure into a new collection of surviving states with its own “effective” ground state and its own “effective” asymptotic degeneracy of states.

### 4.3 The Type II string

Finally, let us briefly discuss the case of the Type II string. For the Type II string, the ground state  $\bar{\chi}_I \chi_I$  is already level-matched; thus, the above argument for the heterotic string no longer applies. In particular, modular invariance and the fact that this state is bosonic would allow this term to appear within either  $Z^{(1)}$  or  $Z^{(3)}$ . However, we are assuming that our Type II theory is tachyon-free at zero temperature; thus this term cannot appear within  $Z^{(1)}$ . The existence or non-existence of the Hagedorn transition in Type II strings thus depends on whether the ground state  $\bar{\chi}_I \chi_I$  appears within  $Z^{(3)}$ .

To phrase this result somewhat differently, recall that have already seen below Eq. (2.14) that any finite-temperature closed string model can be viewed as interpolating between its zero-temperature version (presumed supersymmetric) as  $T \rightarrow 0$  and a different, non-supersymmetric zero-temperature string model as  $T \rightarrow \infty$ . This latter model must be non-supersymmetric because we expect thermal effects to break any supersymmetry which might have existed at zero temperature. Thus, we see that a given Type II string will have a Hagedorn transition if and only if it interpolates between a tachyon-free model as  $T \rightarrow 0$  and a *tachyonic* model as  $T \rightarrow \infty$ . In other words, the existence of a Hagedorn transition depends on whether the  $T \rightarrow \infty$  limit of the Type II string in question is tachyonic or tachyon-free.

Unfortunately, this is a model-dependent issue. In ten dimensions, it is easy to demonstrate that all non-supersymmetric Type II string models are tachyonic; these are the so-called Type 0A and Type 0B strings. Thus, the only possible  $T \rightarrow \infty$  endpoints for the Type IIA/B strings in ten dimensions are tachyonic, leading to the usual Hagedorn transition. However, in lower dimensions, it is possible to construct Type II strings which are non-supersymmetric but tachyon-free. [Such strings are analogues of the ten-dimensional non-supersymmetric heterotic  $SO(16) \times SO(16)$

string.] Thus, it may be possible to construct finite-temperature Type II string models in  $D < 10$  which have such non-supersymmetric, tachyon-free string models as their  $T \rightarrow \infty$  endpoints, and for which the Hagedorn transition would be entirely absent.

## 5 The Hagedorn transition re-identified

In the previous section, we demonstrated that the usual Hagedorn transition is completely absent in the behavior of thermodynamic quantities for all heterotic strings, and possibly for certain Type II strings as well. Indeed, in such cases, one-loop thermodynamic quantities such as the free energy, the internal energy, the entropy, and even the specific heat remain smooth and undisturbed as a function of temperature, and cross the Hagedorn temperature without so much as a ripple.

Does this mean that there is no such thing as a Hagedorn transition in such strings?

In this section, we shall demonstrate that an alternative phase transition often does exist for such strings, but at a higher temperature. Moreover, unlike the traditional Hagedorn transition, this new transition will be completely observable in the behavior of physical, thermodynamic quantities, appearing as an actual divergence or discontinuity in these quantities as a function of temperature. Indeed, as we shall demonstrate, this new phase transition is much weaker than the traditional (first-order) Hagedorn transition, occurring with an order which depends on the spacetime dimension.

Because the origins of this new phase transition are similar to those of the traditional Hagedorn transition, we shall refer to this new phase transition as a “re-identified” Hagedorn transition. However, we believe that in many cases, this new transition is indeed the only true Hagedorn transition that such string theories experience.

### 5.1 Preliminaries

To see how this new phase transition arises, let us begin more generally by determining whether it is *ever* possible for our string thermodynamic quantities to experience divergences or discontinuities. Although some of the discussion in this subsection is based on ideas already presented in previous sections, our purpose here is to provide a top-down, systematic derivation of the conditions for phase transitions in closed-string thermodynamics.

Recall from Sect. 2 that our thermodynamic quantities take the form

$$\int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \tau_2^x \sum_{yz} a_{yz} \bar{q}^y q^z \quad (5.1)$$

where we have expanded  $Z_{\text{string}}$  as a power series in  $q$  and  $\bar{q}$  with temperature-dependent exponents  $(y, z)$ . Note that the modular invariance of  $Z_{\text{string}}$  implies  $y - z \in \mathbb{Z}$ . As a result of the truncation of the region of integration to  $\mathcal{F}$ , there is no ultraviolet divergence from the region  $\tau_2 \rightarrow 0$ . Thus, the only possible divergence is an infrared divergence arising from the region  $\tau_2 \rightarrow \infty$ . In this region, however, the  $\tau_1$ -integration across the entire range  $-1/2 \leq \tau_1 \leq 1/2$  enforces level-matching, eliminating terms in Eq. (5.1) for which  $y \neq z$ . Thus, only terms of the form  $\bar{q}^y q^y$  survive. In the infrared region  $\tau_2 \rightarrow \infty$ , our integral therefore behaves as

$$\int_0^\infty \frac{d\tau_2}{\tau_2^2} \tau_2^x \sum_y a_{yy} \exp(-4\pi\tau_2 y) . \quad (5.2)$$

Of course, we have already seen in Sect. 3 that  $a_{yy} \sim e^{y\sqrt{y}}$  as  $y \rightarrow \infty$ . Thus, the  $y$ -summation in Eq. (5.2) is convergent for all  $\tau_2 > 0$ . In other words, we cannot achieve a divergent integral as a result of the summation over physical string states.

We see, then, that divergences in Eq. (5.2) can arise only if an individual term in Eq. (5.2) is divergent. However, it is immediately clear that all terms with  $y > 0$  are individually convergent for all  $x$ . Thus, depending on the value of  $x$ , divergences can only arise from terms with  $y \leq 0$  — *i.e.*, terms corresponding to massless or tachyonic string states. Of course, this is not a surprise: the appearance of massless states at a certain critical temperature is associated with the emergence of long-range order, which signals the onset of a phase transition. Likewise, a tachyonic state signifies an unstable ground state, thereby triggering the vacuum shift associated with the phase transition. However, we now see that there is, indeed, no other way in which our string thermodynamic quantities can diverge.

This is an important point which we shall again emphasize: the study of a possible Hagedorn transition reduces to a study of the tachyonic and/or massless string states, amounting to an essentially IR analysis. In particular, the usual UV arguments are not sufficiently precise in their standard forms to determine whether or not a divergence arises for the vacuum amplitude. Of course, the UV and IR approaches are related through Poisson resummations and through modular mappings between the fundamental domain  $\mathcal{F}$  and the strip  $\mathcal{S}$ ; thus the two approaches are ultimately equivalent. Our point, however, is that it is the IR analysis which is more direct within the  $\mathcal{F}$ -representation for the vacuum amplitudes.

Comparing with Eq. (4.3), we see that the parameter  $y$  can be identified as  $\frac{1}{4}\alpha' M_{\text{tot}}^2$ , where

$$4y \equiv \alpha' M_{\text{tot}}^2 = 2(\tilde{m} + \tilde{n}) + \frac{m^2 T^2}{\mathcal{M}^2} + \frac{n^2 \mathcal{M}^2}{T^2} . \quad (5.3)$$

Here  $(\tilde{m}, \tilde{n}, m, n)$  respectively represent the zero-temperature right-moving excitation number, the zero-temperature left-moving excitation number, the thermal momentum number, and the thermal winding number of a given string state. Recall that

for physical states in the  $\tau_2 \rightarrow \infty$  region, these quantum numbers are subject to the level-matching constraint

$$\tilde{n} - \tilde{m} + mn = 0. \quad (5.4)$$

Given this, let us classify the different classes of massless and/or tachyonic states. Since the thermal contributions to Eq. (5.3) are necessarily positive, we see that there are only four ways of achieving level-matched massless or tachyonic states:

- We can have  $\tilde{m} = \tilde{n} = 0$ , with  $m = n = 0$ . Such terms correspond to massless states (*e.g.*, the graviton) which appear at zero temperature, and which remain massless at all temperatures. We shall refer to such states as “regular massless” states. All string models contain such states.
- We can have  $\tilde{m} = \tilde{n} < 0$ , with  $n = 0$ ,  $m$  arbitrary (including  $m = 0$ ). These terms represent physical tachyons at zero temperature. As the temperature is increased, the thermal excitations with  $m \neq 0$  eventually become massless and then massive, while the  $m = 0$  state remains tachyonic for all temperatures. Because of their zero-temperature physical tachyons, models containing such terms are uninteresting from a phenomenological perspective, and will not be considered further. In some sense, they are already unstable at zero temperature, and can be expected to undergo zero-temperature phase transitions which are not thermal in nature.
- We can have  $\tilde{m} = \tilde{n} < 0$ , with  $m = 0$ ,  $n \neq 0$ . A model containing such a term is not tachyonic at zero temperature. However, it contains a tower of massive thermal winding states (corresponding to different values of  $n$ ) which become massless at specific temperatures and then tachyonic beyond those temperatures. Formally, these tachyons then persist for all temperatures beyond these critical temperatures, leading to a tachyonic  $T \rightarrow \infty$  endpoint.
- Finally, we can have  $\tilde{m} \neq \tilde{n}$  and  $\tilde{m} + \tilde{n} < 0$ , with both  $m \neq 0$  and  $n \neq 0$ . As long as Eq. (5.4) is satisfied and  $\alpha' M_{\text{tot}}^2 = 0$ , this will also generally represent a massless physical state at two critical temperatures  $T_1$  and  $T_2$ , with  $T_1 \leq T_2$ . [These are the analogues of the temperatures  $T^{(\pm)}$  in Eq. (3.16).] Such a state is then massive for  $T < T_1$  (thanks to the non-zero thermal winding mode), and is also massive for  $T > T_2$  (thanks to the non-zero thermal momentum mode). Between  $T_1$  and  $T_2$ , the state is tachyonic; however if  $\tilde{m} = 0$  or  $\tilde{n} = 0$ , then  $T_1 = T_2$ , thereby eliminating the intermediate tachyonic temperature interval. Note that in all cases, neither the  $T \rightarrow 0$  nor the  $T \rightarrow \infty$  endpoints exhibit physical tachyons.

We shall refer to states in the final category as “thermally massless” at their critical temperatures  $T_{1,2}$  because masslessness is achieved at these specific temperatures as the result of a balance between a “bare” tachyonic mass and an additional non-zero

thermal mass contribution. It is these “thermally massless” states which will be our main focus in the rest of this section.

Note that much of this discussion mirrors the discussion in Sect. 3, where we reviewed the traditional arguments for the Hagedorn transition. As we have seen, the traditional arguments in the Type II case rely on the existence of states with  $(\tilde{m}, \tilde{n}, m, n) = (-1/2, -1/2, 0, \pm 1)$ . Such states are in our third category above, and only exist for those finite-temperature Type II strings whose  $T \rightarrow \infty$  limits are tachyonic. Likewise, the traditional argument in the heterotic case rests upon the existence of massless states with  $(\tilde{m}, \tilde{n}, m, n) = (-1/2, -1, \pm 1/2, \pm 1)$ . Such states would have been in our final “thermally massless” category if they had survived the appropriate GSO projections. However, as we have already discussed, these states generically fail to survive the appropriate GSO projections, and thus do not appear in the one-loop string partition function.

## 5.2 Physical, off-shell tachyons: the “proto-graviton” and “proto-gravitino”

The question that we face, then, is a simple one: *in the heterotic string, what is the new, “effective” string ground state which actually survives the finite-temperature GSO projections?* In other words, what “thermally massless” states actually do exist in the thermal string partition function? Note that in the  $T \rightarrow 0$  limit in which our thermal  $\mathcal{E}/\mathcal{O}$  functions melt away, this ultimately becomes a question about off-shell (but otherwise physical) tachyons in the zero-temperature theory: which off-shell tachyons generically survive the GSO constraints in a given heterotic string model? Of course, we are concerned with off-shell tachyons because we are restricting our attention to states with  $m + n < 0$  but  $m \neq n$ ; as discussed above, on-shell tachyons would have led to instabilities already at zero temperature.

Since we are looking for generic states, let us first consider the only other generic massless states in the heterotic string, namely those associated with the gravity multiplet. Recall that in the heterotic string, the graviton is realized in the Neveu-Schwarz sector as

$$\text{graviton:} \quad g^{\mu\nu} \subset \tilde{b}_{-1/2}^\mu |0\rangle_R \otimes \alpha_{-1}^\nu |0\rangle_L \quad (5.5)$$

where  $\tilde{b}_{-1/2}^\mu$  and  $\alpha_{-1}^\nu$  are respectively the excitations of the right-moving worldsheet Neveu-Schwarz fermion  $\tilde{\psi}^\mu$  and left-moving worldsheet coordinate boson  $X^\nu$ . Since the Neveu-Schwarz heterotic string ground state has vacuum energies  $(H_R, H_L) = (-1/2, -1)$ , as in Eq. (3.10), the states in Eq. (5.5) are both level-matched and massless, with  $(H_R, H_L) = (\tilde{m}, \tilde{n}) = (0, 0)$ . These states include the spin-two graviton, the spin-one antisymmetric tensor field, and the spin-zero dilaton.

In a similar vein, any model exhibiting spacetime supersymmetry must also contain the gravitino state, realized in the Ramond sector of the heterotic string as

$$\text{gravitino:} \quad \tilde{g}^{\alpha\nu} \subset \{\tilde{b}_0\}^\alpha |0\rangle_R \otimes \alpha_{-1}^\nu |0\rangle_L . \quad (5.6)$$

Here  $\{\tilde{b}_0\}^\alpha$  schematically indicates the Ramond zero-mode combinations which collectively give rise to the spacetime Lorentz spinor index  $\alpha$ , as required for the spin-3/2 gravitino state.

Regardless of the particular GSO projections, we know that the graviton state (5.5) must always appear in the string spectrum; likewise, if the model has spacetime supersymmetry, we know that the gravitino state (5.6) must exist as well. However, it is then straightforward to show that this implies that certain additional off-shell tachyons must also exist in the string spectrum. Specifically, regardless of the particular GSO projections, we must always have a spin-one “proto-graviton” state  $\phi^\mu$  in the Neveu-Schwarz sector:

$$\text{proto-graviton: } \phi^\mu \equiv \tilde{b}_{-1/2}^\mu |0\rangle_R \otimes |0\rangle_L ; \quad (5.7)$$

likewise, if the model is spacetime supersymmetric, we must also have a spin-1/2 “proto-gravitino” state  $\psi^\alpha$  in the Ramond sector:

$$\text{proto-gravitino: } \tilde{\phi}^\alpha \equiv \{\tilde{b}_0\}^\alpha |0\rangle_R \otimes |0\rangle_L . \quad (5.8)$$

Note that these are the same states as the graviton/gravitino, except that in each case the left-moving bosonic excitation is lacking. However, it is important to realize that *GSO projections are completely insensitive to the presence or absence of excitations of the worldsheet coordinate bosonic fields*. Thus, since the graviton is always present, it then follows that the proto-graviton must also always be present; likewise, if the model is supersymmetric and the gravitino is present, then the proto-gravitino must also always be present.

While there are many ways to see that the GSO projections must treat the states in the gravity multiplet and their proto-counterparts in exactly the same manner, it is perhaps easiest to understand this fact from modular invariance. The heterotic string partition function in  $D$  dimensions generally takes the form

$$Z_{\text{model}}(\tau) = \tau_2^{1-D/2} \frac{\sum_i \overline{\Theta}_i^{14-D} \Theta_i^{26-D}}{\bar{\eta}^{12} \eta^{24}} \quad (5.9)$$

where the  $\Theta$ -function numerator schematically represents lattice sums over compactified momenta and winding modes, and where the  $\eta$ -function denominator represents the contributions from the bosonic oscillators. While the GSO projections play a role in determining which particular combinations of  $\Theta$ -functions can appear in forming a modular-invariant numerator, the  $\eta$ -function denominators are universally present for all string models and are modular invariant by themselves. Indeed, as evident from Eq. (5.9), they are not even affected by spacetime compactification. Therefore, as long as the graviton and gravitino exist in a given supersymmetric string model, the proto-graviton and proto-gravitino states must also exist because their contributions are all encoded within the same  $\eta$ -denominator regardless of the specific  $\Theta$ -function numerators.

Thus, we conclude that the proto-graviton and proto-gravitino are two off-shell tachyons with worldsheet energies  $(H_R, H_L) = (0, -1)$  which generically appear in all supersymmetric heterotic string models. Moreover, as we shall see, these are often the effective ground states of the heterotic string which necessarily survive after the GSO projections have been applied.

### 5.3 Re-identifying the Hagedorn transition

Let us now consider the physical effects induced by thermal excitations of these proto-graviton and proto-gravitino states. Our first task is to determine the situations in which these states can become “thermally massless” — *i.e.*, the situations in which they can have massless, on-shell thermal excitations.

Our calculation proceeds exactly as in previous sections. The proto-graviton and proto-gravitino each have worldsheet energies  $(H_R, H_L) = (0, -1)$ , and at non-zero temperature their thermal momentum and winding excitations receive additional contributions of the form in Eq. (3.11). Requiring massless, on-shell states with  $(H_R^{\text{tot}}, H_L^{\text{tot}}) = (0, 0)$ , we thus find two solutions at two different temperatures:

$$\begin{cases} m = 1, \quad n = 1, \quad a = 1 & \Rightarrow \text{corresponds to } \mathcal{O}_0, \\ m = 1/2, \quad n = 2, \quad a = 2 & \Rightarrow \text{corresponds to } \mathcal{E}_{1/2}. \end{cases} \quad (5.10)$$

Note that in each case above, we have also indicated which of the  $\mathcal{E}/\mathcal{O}$  thermal sums in Eq. (2.10) would give rise to this contribution.

It is now easy to determine which of these solutions is self-consistent from a physical standpoint. Because of its integer momentum number, the first solution must correspond to a spacetime boson. Thus, it can only apply for the proto-graviton rather than the proto-gravitino. However, just like the graviton state from which it is derived, the proto-graviton should appear within the untwisted sector  $Z^{(1)}$  in Eq. (2.12), not the sector  $Z^{(3)}$  corresponding to  $\mathcal{O}_0$ . Thus, the first solution in Eq. (5.10) cannot be realized for either of our two proto-states. The second solution, by contrast, can only apply for the fermionic proto-gravitino state because of its half-integer momentum number. Fortunately, because this solution corresponds to  $\mathcal{E}_{1/2}$  rather than  $\mathcal{O}_{1/2}$ , it requires the proto-gravitino state to appear exactly where it does appear: within the untwisted fermionic sector  $Z^{(2)}$  rather than the twisted fermionic sector  $Z^{(4)}$ .

Thus, we conclude that for supersymmetric heterotic string models, the proto-gravitino state has a thermal excitation with  $(m, n) = (1/2, 2)$  which becomes “thermally massless” at the temperature  $a \equiv 2\pi T/M_{\text{string}} = 2$ . By contrast, the proto-graviton state does not have any potentially massless thermal excitations.

Because of its non-zero thermal momentum and winding numbers, this thermal excitation of the proto-gravitino state is extremely massive at both small and large temperatures, becoming massless only at the specific temperature  $a = 2$ . Specifically,

we see from Eq. (5.3) that the mass of this state is given by

$$\alpha' M_{\text{tot}}^2 = \frac{a^2}{4} + \frac{4}{a^2} - 2 . \quad (5.11)$$

Note that unlike the case of the traditional  $(m, n) = (\pm 1/2, \pm 1)$  excitation of the heterotic ground state, the  $(1/2, 2)$  excitation of the proto-gravitino state never becomes tachyonic: this state merely hits masslessness at  $a = 2$  before becoming massive again at higher temperatures. Of course, this result is completely consistent with the fact that the proto-gravitino state is fermionic, since the existence of a physical fermionic tachyon at any temperature would violate Lorentz invariance.

However, given that this state never becomes tachyonic, it is natural to wonder whether this state can ever give rise to a Hagedorn transition. Indeed, since no tachyon ever develops, it may appear that our thermal vacuum amplitude  $\mathcal{V}(T)$  will never diverge. It is easy to verify this expectation. In general, the  $(m, n) = (1/2, 2)$  thermal excitation of the proto-gravitino state makes a contribution to  $\mathcal{V}(T)$  given by

$$\begin{aligned} \mathcal{V}(T) &= -\frac{1}{2}\mathcal{M}^{D-1} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \tau_2^{1-D/2} \sqrt{\tau_2} \frac{1}{q} [\bar{q}^{(a/2-2/a)^2/4} q^{(a/2+2/a)^2/4}] + \dots \\ &= -\frac{1}{2}\mathcal{M}^{D-1} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \tau_2^{1-D/2} \sqrt{\tau_2} e^{2\pi\tau_2} e^{-\pi\tau_2(a^2/4+4/a^2)} + \dots \end{aligned} \quad (5.12)$$

where we have left the temperature  $a \equiv 2\pi T/M_{\text{string}}$  arbitrary. Note that the leading  $1/q$  factor in the first line of Eq. (5.12) represents the zero-temperature contribution from the proto-gravitino, with  $(H_R, H_L) = (0, -1)$ , while the remaining factor in brackets represents the thermal contribution with  $(m, n) = (1/2, 2)$ . Likewise, we have carefully recorded all factors of  $\tau_2 \equiv \text{Im } \tau$ : we have two factors of  $\tau_2$  in the denominator from the modular-invariant measure of integration; we have  $(1 - D/2)$  factors in the numerator from the zero-temperature partition function  $Z_{\text{model}}$  in Eqs. (2.5) and (5.9); and we have an additional factor  $\sqrt{\tau_2}$  in the numerator from the Matsubara thermal sums in Eq. (2.8). However, at  $a = 2$ , this expression reduces to

$$\mathcal{V}(T) \Big|_{a=2} = -\frac{1}{2}\mathcal{M}^{D-1} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \tau_2^{1-D/2} \sqrt{\tau_2} + \dots \quad (5.13)$$

and as  $\tau_2 \rightarrow \infty$ , this contribution scales like

$$\int^{\infty} \frac{d\tau_2}{\tau_2^{(1+D)/2}} . \quad (5.14)$$

This contribution is therefore finite for all  $D \geq 2$ . This, of course, agrees with our usual expectation that a massless state does not lead to a divergent vacuum amplitude in two or more spacetime dimensions.

However, let us now investigate temperature derivatives of  $\mathcal{V}(T)$ . As evident from the second line of Eq. (5.12), each temperature derivative  $d/dT \sim d/da$  brings down an extra factor of  $\tau_2$ . In general, this thereby *increases the tendency towards divergence of our thermodynamic quantities*.

Our results are as follows. The contribution of this thermally excited proto-gravitino state to the first derivative  $d\mathcal{V}/da$  is given by

$$\frac{d\mathcal{V}}{da} = \pi \mathcal{M}^{D-1} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \tau_2^{1-D/2} \sqrt{\tau_2} \tau_2 \left( \frac{a}{4} - \frac{4}{a^3} \right) e^{2\pi\tau_2} e^{-\pi\tau_2(a^2/4+4/a^2)} + \dots, \quad (5.15)$$

but at the temperature  $a = 2$  we see that the factor in parenthesis within Eq. (5.15) actually vanishes:

$$\left. \frac{d\mathcal{V}}{da} \right|_{a=2} = 0. \quad (5.16)$$

It turns out that this is a general property, reflecting nothing more than the fact that the slope of the mass function in Eq. (5.11) vanishes at its minimum, as it must. However, taking subsequent derivatives and evaluating at  $a = 2$ , we find the general pattern

$$\left. \frac{d^p \mathcal{V}}{da^p} \right|_{a=2} = \mathcal{M}^{D-1} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \tau_2^{1-D/2} \sqrt{\tau_2} f_p(\tau_2) + \dots \quad (5.17)$$

where  $f_p(\tau_2)$  for  $p \geq 2$  is a rank- $r$  polynomial in  $\tau_2$  of the form

$$f_p(\tau_2) = A_p \tau_2^r + B_p \tau_2^{r-1} + C_p \tau_2^{r-2} \dots, \quad (5.18)$$

where

$$r = \begin{cases} p/2 & \text{for } p \text{ even} \\ (p-1)/2 & \text{for } p \text{ odd}, \end{cases} \quad (5.19)$$

and where the leading coefficients  $A_p$  are positive for  $p = 1, 2 \pmod{4}$  and negative for  $p = 0, 3 \pmod{4}$ , with alternating signs for the lower-order coefficients  $B_p, C_p$ , etc. Given these extra leading powers of  $\tau_2$ , we thus find that as a result of the proto-gravitino state,

$$\frac{d^p \mathcal{V}}{dT^p} \quad \text{diverges for} \quad \begin{cases} D \leq p & \text{for } p \text{ odd} \\ D \leq p+1 & \text{for } p \text{ even.} \end{cases} \quad (5.20)$$

Equivalently, in  $D \geq 2$  spacetime dimensions, the proto-gravitino state results in a divergence that first occurs for  $d^p \mathcal{V}/dT^p$ , where

$$p = \begin{cases} D & \text{for } D \text{ even} \\ D-1 & \text{for } D \text{ odd.} \end{cases} \quad (5.21)$$

This divergence then corresponds to a new, re-identified, Hagedorn phase transition.

We stress that it is not merely the masslessness of this thermally-enhanced proto-gravitino state that results in this phase transition. It is the fact that this masslessness

is achieved *thermally*, with non-trivial thermal momentum and winding quanta, that induces this phase transition. By contrast, a regular massless state such as the usual graviton or gravitino does not contribute to any temperature derivatives of  $\mathcal{V}$ .

To interpret the physical ramifications of this new Hagedorn phase transition, we recall that the free energy  $F$  scales with  $\mathcal{V}$  itself, while the internal energy  $U$  and entropy  $S$  involve  $d\mathcal{V}/dT$ , the specific heat  $c_V$  involves  $d^2\mathcal{V}/dT^2$ , and subsequent temperature derivatives of  $c_V$  involve higher derivatives of  $\mathcal{V}$ . We thus see that  $F$  never diverges (it would formally diverge only for  $D \leq 1$ ), while  $U$  and  $S$  also never diverge for *any* spacetime dimensions because the first derivative  $d\mathcal{V}/dT$  is always finite, as shown in Eq. (5.16). The finiteness of  $U$  at the critical temperature suggests that this is not a limiting temperature, but only the location of a phase transition. Likewise,  $c_V$  and  $dc_V/dT$  diverge only for  $D \leq 3$ , while  $d^2c_V/dT^2$  diverges for  $D \leq 5$ , etc.

Thus, for heterotic strings in  $D$  dimensions, we conclude that the proto-gravitino *does* give rise to a Hagedorn transition. The associated Hagedorn temperature is  $a = 2$  as opposed to the traditional value  $a = 2 - \sqrt{2}$ . Moreover, this re-identified phase transition is generally a very weak  $p^{\text{th}}$ -order phase transition, where  $p$  is given in Eq. (5.21). However, unlike the traditional Hagedorn transition, this re-identified Hagedorn transition leaves a *bona-fide* imprint in the behavior of string thermodynamic quantities. In particular, for  $D = 4$ , this is a fourth-order phase transition in which  $d^2c_V/dT^2$  diverges, causing  $dc_V/dT$  to experience a discontinuity, the specific heat  $c_V$  itself to experience a kink, and the internal energy function to have a discontinuous change in curvature.

## 6 Other Hagedorn transitions

In the previous section, we analyzed the Hagedorn transition induced by the heterotic proto-gravitino state. As we discussed, this off-shell tachyonic state must always exist in a supersymmetric heterotic string model, regardless of the spacetime dimension. Thus, the Hagedorn transition we found is completely generic within the class of supersymmetric heterotic string models.

Depending on the particular model under study, however, there may be other off-shell (or even on-shell) tachyons whose thermal excitations can also give rise to Hagedorn-like transitions. Clearly, only the transition that occurs with the lowest temperature will be the “true” Hagedorn transition; beyond this temperature, the degrees of freedom of the system can change in a way that eliminates or modifies all further transitions. Thus, for completeness, it is also important to study the other off-shell (and even on-shell) tachyons which may exist in such models, and the corresponding Hagedorn transitions which can potentially arise.

Unlike the situation with the proto-graviton and proto-gravitino, the tachyonic structure of a given string model is highly model-dependent. The relevant vacuum energies that may emerge depend crucially on the particular GSO projections, orbifold

twists, and the like. However, for concreteness, we shall here assume only half-integer vacuum energies. Within a heterotic string whose ground state vacuum energies are  $(H_R, H_L) = (-1/2, -1)$ , this then gives rise to only four classes of possible tachyons, grouped by their vacuum energies:  $(H_R, H_L) = (-1/2, -1)$ ,  $(0, -1)$ ,  $(0, -1/2)$ , and  $(-1/2, -1/2)$ . We shall discuss each of these cases in turn. As a check on our classification, we note that there exists a duality symmetry

$$a \rightarrow 2/a \quad (6.1)$$

under which we merely exchange the roles of  $\mathcal{O}_0$  and  $\mathcal{E}_{1/2}$ . Thus, within each class of tachyon vacuum energies, the solutions for the corresponding potential Hagedorn temperatures should come in dual pairs related by Eq. (6.1).

The tachyons in the first class, with  $(H_R, H_L) = (-1/2, -1)$ , are the usual ground-state tachyons of the heterotic string. As we have discussed in Sect. 3, these off-shell tachyons would seem to have a massless thermal excitation with  $(m, n) = (1/2, 1)$  (corresponding to  $\mathcal{O}_{1/2}$ ), which would in turn give rise to the traditional Hagedorn transition at  $a = 2 - \sqrt{2}$ . (This thermal mode would also be massless at the dual temperature  $a = 2 + \sqrt{2}$ , and tachyonic between these two temperatures). However, as we have discussed in Sect. 4, such states are GSO-projected out of the spectrum, thereby eliminating the corresponding (traditional) Hagedorn divergence.

The tachyons in the second class, with  $(H_R, H_L) = (0, -1)$ , are the proto-gravity states we introduced in Sect. 5. As we discussed, although the proto-graviton state with thermal excitation  $(m, n) = (1, 1)$  would seem to lead to a Hagedorn transition at temperature  $a = 1$ , the  $\mathbb{Z}_2$  thermal orbifold twist actually eliminates this thermal excitation, requiring only  $m \in 2\mathbb{Z}$  (*i.e.*, forcing this state to arise in the  $\mathcal{E}_0$  sector rather than the  $\mathcal{O}_0$  sector). Thus, the proto-graviton does not give rise to a Hagedorn transition. By contrast, as we discussed in Sect. 5, the proto-gravitino has an allowed massless thermal excitation with  $(m, n) = (1/2, 2)$  giving rise to a Hagedorn transition at the dual temperature  $a = 2$ . This, then, is our re-identified Hagedorn transition. Of course, the proto-gravitino state exists only if our string model has spacetime supersymmetry at zero temperature.

This much has already been discussed in previous sections. Let us now turn to the tachyons in the third class, with  $(H_R, H_L) = (0, -1/2)$ . These off-shell states have a massless thermal excitation  $(m, n) = (1/2, 1)$  which would give rise to a Hagedorn transition at the self-dual temperature  $a = \sqrt{2}$ . If this is to occur, such states must clearly arise in the twisted  $\mathcal{O}_{1/2}$  sector [*i.e.*, within  $Z^{(4)}$  in Eq. (2.12)]. Of course, whether these states exist in such a twisted sector is clearly a model-dependent issue. However, if they do exist, their  $(m, n) = (1/2, 1)$  thermal excitations will be massive for all temperatures other than  $a = \sqrt{2}$ . Thus, the corresponding Hagedorn transition in this case will be a weak one whose order, like that induced by the proto-gravitino, depends on the spacetime dimension through Eq. (5.21). Note that such tachyons can in principle appear not only for heterotic strings, but also for Type II strings.

Finally, we turn to the  $(H_R, H_L) = (-1/2, -1/2)$  tachyons. Once again, such tachyonic states can arise within both Type II and heterotic strings. For this (on-shell) vacuum energy configuration, there are four different cases which must be considered:

- First, such tachyons can give rise to massless thermal modes of the form  $(m \in \mathbb{Z}, n = 0)$ , corresponding to  $\mathcal{E}_0$ . For  $m \neq 0$ , such thermal excitations are massless at  $a = \sqrt{2}/|m|$ , massive above this temperature, but tachyonic below it. Models containing such states would therefore contain physical, on-shell tachyons even at zero temperature [this is the non-thermal  $(m, n) = (0, 0)$  tachyon itself], and presumably experience non-thermal phase transitions already at zero temperature. Models containing such states are therefore beyond our consideration.
- Second, such tachyons can give rise to massless thermal modes of the form  $(m = 0, n \in 2\mathbb{Z})$ , again corresponding to  $\mathcal{E}_0$ . For  $n \neq 0$ , such thermal excitations are massless at the dual temperature  $a = |n|/\sqrt{2}$ , massive below this temperature, and tachyonic above it. However, because these states must multiply  $\mathcal{E}_0$ , they can only arise in a model which also contains the  $(m = 0, n = 0)$  non-thermal excitation. This is a non-thermal state which is tachyonic at all temperatures, including zero. Therefore, as in the case above, models containing such states are beyond our consideration.
- Third, such states could in principle give rise to massless thermal modes of the form  $(m \in \mathbb{Z} + 1/2, n = 0)$ , corresponding to  $\mathcal{E}_{1/2}$ . As in the first case above, such thermal excitations would be massless at  $a = \sqrt{2}/|m|$ , massive above this temperature, and tachyonic below it. However, such states can never arise, since their non-integer thermal momentum number indicates that they must be spacetime fermions. Lorentz invariance precludes the existence of fermionic tachyons. (In string language, this happens because the Ramond zero-mode vacuum is never tachyonic.)
- Finally, such states can give rise to massless thermal modes of the form  $(m = 0, n \in 2\mathbb{Z} + 1)$ , corresponding to  $\mathcal{O}_0$ . Such thermal excitations are massless at the dual temperature  $a = |n|/\sqrt{2}$ , massive below this temperature, and tachyonic above it. These models are thus tachyon-free at zero temperature, but have physical, on-shell tachyons as  $T \rightarrow \infty$ . Such models are not inconsistent. Indeed, the zero-temperature Type II strings in  $D = 10$  are in this class, with the  $(m, n) = (0, \pm 1)$  modes giving rise to the traditional Type II Hagedorn transition at  $a = 1/\sqrt{2}$ . However, even for Type II strings, tachyons in this class are not *guaranteed* to exist. In dimensions  $D < 10$ , for example, there exist Type II models which are non-supersymmetric but tachyon-free [analogues of the ten-dimensional  $SO(16) \times SO(16)$  heterotic string model]. These models

could potentially serve as  $T \rightarrow \infty$  endpoints of a finite-temperature interpolation away from a supersymmetric zero-temperature limit, thereby evading the traditional Hagedorn transition for such Type II models.

We see, then, that the particular Hagedorn transitions that a given string model may experience are closely tied to the on-shell and off-shell tachyonic vacuum structure of the spectrum. This in turn depends not only on the original zero-temperature model in question, but also on the specific interpolation that represents its behavior at finite temperature.

## 7 Discussion, speculations, and extensions

In this paper, we have critically re-evaluated the Hagedorn transition within the context of closed, perturbative Type II and heterotic strings. As we discussed, the details of the Hagedorn transition turn out to depend critically on the precise manner in which a given zero-temperature string theory is extrapolated to finite temperature.

For broad classes of closed string theories, we found that the traditional Hagedorn transition is completely absent when the correct extrapolation is used. This is the case for all heterotic strings, and also potentially for certain Type II strings in  $D < 10$ . Indeed, within these classes of string theories, the usual Hagedorn phase transition leaves no imprint whatsoever in the behavior of one-loop thermodynamic quantities such as the string free energy, the internal energy, or the entropy. As we explained, the usual Hagedorn transition is eliminated in such cases as a result of the GSO projections that are necessary in order to produce self-consistent string models at finite temperature.

However, and potentially more importantly, we found that there is an alternative “re-identified” Hagedorn transition for heterotic strings which is triggered by the thermal winding excitations of a different, “effective” tachyonic string ground state. Unlike the usual CFT ground state which is eliminated by the GSO projections in the heterotic case, this new “effective” string ground state is physical, and serves as the *bona-fide* ground state of the theory *after* the GSO projections have been applied and after various towers of string states have been removed. These new effective string ground states (corresponding, in some cases, to the proto-gravitino states discussed in Sect. 5) are not as deeply tachyonic as the original worldsheet CFT ground states, and consequently their thermal excitations give rise to Hagedorn-like phase transitions at higher temperatures than expected. Nevertheless, in the absence of the usual Hagedorn transitions, we believe that these new “re-identified” phase transitions are the only Hagedorn transitions that such strings experience.

Let us therefore summarize what we believe to be the final status regarding the (non-)existence of the Hagedorn transition for closed strings.

For heterotic strings, we believe that the usual Hagedorn transition does not exist. Indeed, its existence would require the worldsheet CFT ground state to appear within

the  $Z^{(4)}$  sector of the theory, as indicated in Eq. (2.12), yet this state should only appear within the untwisted sector. Thus, barring any counterexamples to this assertion (and we are not aware of any such special cases), the usual Hagedorn transition cannot exist. We shall illustrate this through explicit examples in Appendix A. Nevertheless, in certain cases, a “re-identified” Hagedorn phase transition may exist. For heterotic string models which are supersymmetric at zero temperature, we showed that there must always exist a re-identified transition at higher temperature whose order depends on the spacetime dimension. However, there may also be other Hagedorn phase transitions which arise at even lower temperatures, and which would take priority; the existence of these other Hagedorn transitions is thus a model-dependent question. This will be discussed further in Appendix A.

For Type II strings, the situation is even more model-dependent. As with the heterotic case, the issue boils down to the construction of self-consistent modular-invariant finite-temperature interpolating models with partition functions of the form in Eq. (2.12). Such partition functions necessarily interpolate between the original model at  $T = 0$  and a second model which serves as the  $T \rightarrow \infty$  limit of the interpolation. Since thermal effects necessarily break supersymmetry, the only finite-temperature partition functions which can describe the thermal behavior of the Type IIA and Type IIB strings are those that interpolate between these strings at  $T = 0$  and the Type 0A and/or Type 0B strings at  $T = \infty$ . Since the latter strings have  $(H_R, H_L) = (-1/2, -1/2)$  tachyons, and since these tachyons are indeed the superstring CFT ground states, the ten-dimensional Type IIA and Type IIB strings indeed experience usual Hagedorn transitions. However, this need not hold in  $D < 10$ . Specifically, for  $D < 10$  there exist superstring (Type II) models which are non-supersymmetric but tachyon-free; these are analogues of the ten-dimensional  $SO(16) \times SO(16)$  heterotic string. If such models can be exploited as the  $T \rightarrow \infty$  endpoints of finite-temperature interpolations away from zero-temperature supersymmetric Type II models, such models would evade the usual Hagedorn transition. This again is a model-dependent question.

We stress that in cases where our re-identified Hagedorn transition is triggered by thermal modes that become tachyonic, none of the standard interpretations of this transition need to be modified. Whether interpreted as a phase transition to a long string state or as a breakdown of the string worldsheet, our re-identification merely implies a new temperature for this phase transition and a different thermal state as its trigger. On the other hand, the re-identified Hagedorn transition associated with the proto-gravitino state is always a weak one whose order depends on the spacetime dimension. Indeed, the appropriate thermal excitation of the proto-gravitino state never becomes tachyonic; it simply becomes massless at a single, critical temperature. In this case, the physical properties of the corresponding phase transition are undoubtedly different.

Finally, let us briefly mention the case of Type I strings. Of course, our analysis in this paper has focused on only the closed-string sectors; thus our results should

extend directly to the closed-string sectors of the Type I strings. However, the open-string sectors are beyond the analysis we have performed. In particular, the absence of modular invariance and the emergence of the ultraviolet  $t \rightarrow 0$  limit of integration for open-string amplitudes suggests that a Hagedorn transition is quite likely to occur in such sectors. On the other hand, it is possible that tadpole anomaly constraints can conspire to eliminate the Hagedorn divergences from such sectors as well, in analogy with the manner in which modular invariance can eliminate the Hagedorn divergences from closed-string sectors. This issue clearly requires further study [12, 35, 36].

Of course, our analysis in this paper is subject to a number of important caveats. First, we are dealing with one-loop string vacuum amplitudes, and likewise considering only the tree-level (non-interacting) particle spectrum. Thus, we are neglecting all sorts of particle interactions. Gravitational effects, in particular, can be expected to change the spectrum quite dramatically, and have recently been argued to eliminate the Hagedorn transition by deforming the resulting spectrum away from the expected exponential rise in the degeneracy of states. However, the purpose of this paper has been to show that even in the non-interacting theory which has been regarded for nearly two decades as the classic Hagedorn system, this transition simply does not arise in the expected way, if at all. This therefore casts further doubt on the existence of the Hagedorn transition after interactions are added.

Our results also raise a number of interesting questions. For example, it would be interesting to understand the phenomenological consequences of our observations within the recent brane-world scenarios. In particular, it seems very strange to contemplate a situation in which the bulk (closed-string) and brane (open-string) sectors give rise to different thermodynamic behaviors at finite temperatures, with phase transitions occurring in one sector but not the other. Even more fundamentally, one might also consider the ramifications of these results for the existence of strong/weak coupling dualities at finite temperatures. If our results are correct, it would be extremely important to reconcile the co-existence of Type I strings which exhibit Hagedorn transitions at a certain critical temperature with heterotic strings (their supposed strong-coupling duals) which either fail to exhibit any Hagedorn transitions or exhibit only very high-order phase transitions at very high temperatures. Likewise, it would also be interesting to extend our results to non-flat backgrounds in order to address important questions such as the thermodynamics of black holes, the AdS/correspondence, and so forth. In a similar vein, it would also be interesting to understand the interplay between these results and recent studies of thermal duality [7, 28, 8, 32, 33], especially as far as new phase transitions are concerned.

Another potentially important line of inquiry might be to interpret these results within the context of a so-called “misaligned supersymmetry” [31]. Misaligned supersymmetry is a general phenomenon which describes the spectrum of any non-supersymmetric tachyon-free closed string theory, including a supersymmetric string theory at finite temperature. One of the implications of misaligned supersymmetry is that while the asymptotic degeneracy of bosonic or fermionic states grows expo-

nentially with an exponent corresponding to the traditional Hagedorn temperature, the actual distribution of bosonic and fermionic states experiences a rapid fluctuation between bosonic and fermionic surpluses as one proceeds to higher and higher mass levels. This surprising fluctuation is the manner in which string amplitudes manage to remain finite, even without supersymmetry [31, 37]. However, this rapid fluctuation induces a cancellation such that a new quantity, a so-called “sector-averaged” density of states, experiences only a much *slower* exponential growth [31]. Indeed, as shown in Ref. [31], this reduced rate of exponential growth is directly correlated with the existence of the proto-graviton and proto-gravitino states which (as we have seen in Sect. 5) give rise to our re-identified Hagedorn transition with a re-identified Hagedorn temperature. Thus the true UV manifestation of both the elimination of the traditional Hagedorn transition and the emergence of a re-identified one at higher temperatures is likely to be a misaligned supersymmetry in the asymptotic degeneracy of states. This too should be further explored.

We see, then, that the issue of the Hagedorn phenomenon in string theory is a subtle one which depends quite crucially on the manner in which a zero-temperature string theory is extrapolated to finite temperature. As such, the Hagedorn temperature — and indeed the entire existence or non-existence of the Hagedorn transition — becomes a highly model-dependent question. This, perhaps, is the most significant lesson to be taken from these results.

## Acknowledgments

This work is supported in part by the National Science Foundation under Grant PHY/0301998, by the Department of Energy under Grant DE-FG02-04ER-41298, and by a Research Innovation Award from Research Corporation. We wish to thank S. Abel, J. Erlich, H. Firouzjahi, E. Poppitz, H. Tye, and especially E. Dudas for discussions. We would also like to thank S. Chaudhuri for explaining the contents of Ref. [33]. One of us (KRD) would like to acknowledge the hospitality of the Aspen Center for Physics, and thank the Laboratory for Elementary Particle Physics at Cornell University for hospitality during a visit in November 2004 during which this work was completed and presented.

## Appendix A

In the main body of this paper, we showed that the usual Hagedorn transition fails to appear for all heterotic strings, and we laid out a set of possible re-identified Hagedorn transitions which can generally emerge to replace it. In the case of heterotic strings which are supersymmetric at zero temperature, we showed that there always exists a weak, high-order re-identified phase transition triggered by a thermal excitation of the proto-gravitino state. However, as we noted, this need not necessar-

ily be the ultimate Hagedorn temperature that such models experience, for another tachyonic state could give rise to a different Hagedorn transition at an even lower temperature, thus dominating the thermodynamics. The question as to whether this might occur is unfortunately model-dependent.

In this Appendix, we shall address this question explicitly by analyzing the specific cases of the three tachyon-free heterotic string models in ten dimensions. Along the way, we shall also discuss what we believe constitute self-consistent finite-temperature extrapolations of zero-temperature string models. In this Appendix, we shall focus on the supersymmetric  $SO(32)$  and  $E_8 \times E_8$  heterotic strings as well as the non-supersymmetric  $SO(16) \times SO(16)$  string. In each case, we shall demonstrate that the possibility of additional Hagedorn transitions is indeed realized, and that in each case there is an alternative Hagedorn transition which occurs not at  $a = 2$ , but at  $a = 1/\sqrt{2}$ . Note that this is also the temperature of the (traditional) Hagedorn transition that emerges for Type IIA and Type IIB strings. Thus, we shall see that *all tachyon-free closed string models in ten dimensions have a universal Hagedorn temperature*. We do not know whether this property persists to lower dimensions, as this continues to be a model-dependent question which rapidly becomes more complicated after compactification.

Let us begin our analysis by focusing on the  $D = 10$  supersymmetric  $SO(32)$  heterotic string. At zero temperature, this model can be described by the partition function

$$Z_{SO(32)} = Z_{\text{boson}}^{(8)} (\bar{\chi}_V - \bar{\chi}_S) (\chi_I^2 + \chi_V^2 + \chi_S^2 + \chi_C^2) \quad (\text{A.1})$$

where the contribution from the worldsheet bosons is given as

$$Z_{\text{boson}}^{(n)} \equiv \tau_2^{-n/2} (\bar{\eta}\eta)^{-n}, \quad (\text{A.2})$$

where the contributions from the right-moving worldsheet fermions are written in terms of the barred characters  $\bar{\chi}_7$  of the transverse  $SO(8)$  Lorentz group, and where the contributions from the left-moving (internal) worldsheet fermions are written as products of the unbarred characters  $\chi_i$  of an  $SO(16)$  gauge group. The subscripts  $I$ ,  $V$ ,  $S$ , and  $C$  generally refer to the identity, vector, spinor, and conjugate spinor representations of the  $SO(2n)$  gauge group; these representations have conformal dimensions  $\{h_I, h_V, h_S, h_C\} = \{0, 1/2, n/8, n/8\}$ , and have corresponding characters which can be expressed in terms of Jacobi  $\vartheta$ -functions as

$$\begin{aligned} \chi_I &= \frac{1}{2} (\vartheta_3^n + \vartheta_4^n)/\eta^n = q^{h_I-c/24} (1 + n(2n-1)q + ...) \\ \chi_V &= \frac{1}{2} (\vartheta_3^n - \vartheta_4^n)/\eta^n = q^{h_V-c/24} (2n + ...) \\ \chi_S &= \frac{1}{2} (\vartheta_2^n + \vartheta_1^n)/\eta^n = q^{h_S-c/24} (2^{n-1} + ...) \\ \chi_C &= \frac{1}{2} (\vartheta_2^n - \vartheta_1^n)/\eta^n = q^{h_C-c/24} (2^{n-1} + ...) \end{aligned} \quad (\text{A.3})$$

where the central charge is  $c = n$  at affine level one. Even though the spinor and conjugate spinor representations are distinct, we find that  $\chi_S = \chi_C$  (due to the fact

that  $\vartheta_1 = 0$ ). For the ten-dimensional transverse Lorentz group  $SO(8)$ , the distinction between  $S$  and  $C$  is equivalent to relative spacetime chirality. Note that the  $SO(8)$  transverse Lorentz group has a triality symmetry under which the vector and spinor representations are indistinguishable. Thus  $\bar{\chi}_V = \bar{\chi}_S$ , resulting in a (vanishing) supersymmetric partition function in Eq. (A.1).

This much is standard. However, in order to understand how this ten-dimensional model behaves at finite temperature, we must construct an appropriate nine-dimensional string model which is capable of representing this ten-dimensional string at finite temperature. As we discussed in Sect. 2, such a nine-dimensional model must interpolate between the supersymmetric  $SO(32)$  heterotic string at  $T \rightarrow 0$  (analogous to  $R \rightarrow \infty$ , where  $R$  is the compactification radius for the thermal dimension), and another (presumably non-supersymmetric) ten-dimensional heterotic string model as  $T \rightarrow \infty$  (or  $R \rightarrow 0$ ). In ten dimensions, the space of non-supersymmetric heterotic string models is extremely limited: a complete classification [38] shows that there are only seven such non-supersymmetric models. These are:

- a tachyon-free  $SO(16) \times SO(16)$  model [24, 25];
- a tachyonic  $SO(32)$  model [25, 39];
- a tachyonic  $SO(8) \times SO(24)$  model [25];
- a tachyonic  $U(16)$  model [25];
- a tachyonic  $SO(16) \times E_8$  model [25, 39];
- a tachyonic  $(E_7)^2 \times SU(2)^2$  model [25]; and
- a tachyonic  $E_8$  model [38].

(In all but the last case, the gauge symmetries are realized at affine level one.) Note that only the first of these seven models is devoid of physical tachyons. The remaining six models all have on-shell tachyons at  $(H_R, H_L) = (-1/2, -1/2)$ .

Thus, there are only seven nine-dimensional interpolating models which can potentially represent the thermodynamics of the supersymmetric  $SO(32)$  string, depending on which of the above models is chosen as the  $T \rightarrow \infty$  limit. However, it turns out that not all of these nine-dimensional models actually exist; as expected from the  $\mathbb{Z}_2$  nature of the thermal orbifold, we can build self-consistent nine-dimensional interpolating models only when the  $T \rightarrow \infty$  endpoint model is a  $\mathbb{Z}_2$  orbifold of the original zero-temperature supersymmetric  $SO(32)$  model.

This provides a significant constraint on the remaining possibilities. One possibility, for example, is to construct a nine-dimensional model interpolating between the supersymmetric  $SO(32)$  string and the *non-supersymmetric*  $SO(32)$  string. Note

that the non-supersymmetric  $SO(32)$  string has the partition function

$$Z = Z_{\text{boson}}^{(8)} \times \left\{ \begin{array}{l} \bar{\chi}_I (\chi_I \chi_V + \chi_V \chi_I) + \bar{\chi}_V (\chi_I^2 + \chi_V^2) \\ - \bar{\chi}_S (\chi_S^2 + \chi_C^2) - \bar{\chi}_C (\chi_S \chi_C + \chi_C \chi_S) \end{array} \right\}. \quad (\text{A.4})$$

Using standard techniques described in Ref. [27, 40, 41, 35, 36], it is then straightforward to construct the unique, self-consistent nine-dimensional string model that interpolates between Eqs. (A.1) and (A.4). Basically, we compactify the  $SO(32)$  string on a (thermal) circle, and then orbifold by  $\mathcal{T}Q$  where  $\mathcal{T}$  is a shift of half the circumference around the thermal circle and where  $Q$  is the  $\mathbb{Z}_2$  orbifold that produces the ten-dimensional non-supersymmetric  $SO(32)$  string from the ten-dimensional supersymmetric  $SO(32)$  string. In this case,  $Q$  is nothing but  $(-1)^F$  where  $F$  represents spacetime fermion number; it is in this manner that  $Q$  breaks spacetime supersymmetry. This results in a nine-dimensional interpolating model with the partition function [35]

$$\begin{aligned} Z_A = Z_{\text{boson}}^{(7)} \times \{ & \mathcal{E}_0 [\bar{\chi}_V (\chi_I^2 + \chi_V^2) - \bar{\chi}_S (\chi_S^2 + \chi_C^2)] \\ & + \mathcal{E}_{1/2} [\bar{\chi}_V (\chi_S^2 + \chi_C^2) - \bar{\chi}_S (\chi_I^2 + \chi_V^2)] \\ & + \mathcal{O}_0 [\bar{\chi}_I (\chi_I \chi_V + \chi_V \chi_I) - \bar{\chi}_C (\chi_S \chi_C + \chi_C \chi_S)] \\ & + \mathcal{O}_{1/2} [\bar{\chi}_I (\chi_S \chi_C + \chi_C \chi_S) - \bar{\chi}_C (\chi_I \chi_V + \chi_V \chi_I)] \} \end{aligned} \quad (\text{A.5})$$

Note, in particular, that this reproduces Eq. (A.1) in the  $T \rightarrow 0$  limit as well as Eq. (A.4) in the  $T \rightarrow \infty$  limit. Comparing Eq. (A.5) with Eq. (2.12), it is easy to read off the particular components  $Z^{(1,2,3,4)}$ .

Another possibility might be to build an interpolation between the supersymmetric  $SO(32)$  string and the tachyon-free  $SO(16) \times SO(16)$  string. The latter string has partition function

$$\begin{aligned} Z = Z_{\text{boson}}^{(8)} \times \left\{ & \bar{\chi}_I (\chi_V \chi_C + \chi_C \chi_V) + \bar{\chi}_V (\chi_I^2 + \chi_S^2) \\ & - \bar{\chi}_S (\chi_V^2 + \chi_C^2) - \bar{\chi}_C (\chi_I \chi_S + \chi_S \chi_I) \right\}. \end{aligned} \quad (\text{A.6})$$

The corresponding nine-dimensional interpolating model then has the partition function [40, 35]

$$\begin{aligned} Z_B = Z_{\text{boson}}^{(7)} \times \{ & \mathcal{E}_0 [\bar{\chi}_V (\chi_I^2 + \chi_S^2) - \bar{\chi}_S (\chi_V^2 + \chi_C^2)] \\ & + \mathcal{E}_{1/2} [\bar{\chi}_V (\chi_V^2 + \chi_C^2) - \bar{\chi}_S (\chi_I^2 + \chi_S^2)] \\ & + \mathcal{O}_0 [\bar{\chi}_I (\chi_V \chi_C + \chi_C \chi_V) - \bar{\chi}_C (\chi_I \chi_S + \chi_S \chi_I)] \\ & + \mathcal{O}_{1/2} [\bar{\chi}_I (\chi_I \chi_S + \chi_S \chi_I) - \bar{\chi}_C (\chi_V \chi_C + \chi_C \chi_V)] \} \end{aligned} \quad (\text{A.7})$$

Note that in this case the orbifold  $Q$  also involves a non-trivial Wilson line which is responsible for breaking the gauge group.

As a final example, we can also construct a nine-dimensional interpolation between the supersymmetric  $SO(32)$  string and the non-supersymmetric  $SO(8) \times SO(24)$  string. Letting  $\bar{\chi}$ ,  $\chi$ , and  $\tilde{\chi}$  respectively represent the characters of the right-moving  $SO(8)$  Lorentz group, the left-moving internal  $SO(8)$  gauge group, and the left-moving internal  $SO(24)$  gauge group, we recall that the partition function of the ten-dimensional  $SO(8) \times SO(24)$  string is given by

$$\begin{aligned} Z = & \bar{\chi}_I (\chi_I \tilde{\chi}_S + \chi_S \tilde{\chi}_I) + \bar{\chi}_V (\chi_I \tilde{\chi}_I + \chi_S \tilde{\chi}_S) \\ & - \bar{\chi}_S (\chi_V \tilde{\chi}_V + \chi_C \tilde{\chi}_C) - \bar{\chi}_C (\chi_V \tilde{\chi}_C + \chi_C \tilde{\chi}_V). \end{aligned} \quad (\text{A.8})$$

This then leads to a unique nine-dimensional interpolating model with the modular-invariant partition function [35]

$$\begin{aligned} Z_C = Z_{\text{boson}}^{(7)} \times \{ & \mathcal{E}_0 [\bar{\chi}_V (\chi_I \tilde{\chi}_I + \chi_S \tilde{\chi}_S) - \bar{\chi}_S (\chi_V \tilde{\chi}_V + \chi_C \tilde{\chi}_C)] \\ & + \mathcal{E}_{1/2} [\bar{\chi}_V (\chi_V \tilde{\chi}_V + \chi_C \tilde{\chi}_C) - \bar{\chi}_S (\chi_I \tilde{\chi}_I + \chi_S \tilde{\chi}_S)] \\ & + \mathcal{O}_0 [\bar{\chi}_I (\chi_I \tilde{\chi}_S + \chi_S \tilde{\chi}_I) - \bar{\chi}_C (\chi_V \tilde{\chi}_C + \chi_C \tilde{\chi}_V)] \\ & + \mathcal{O}_{1/2} [\bar{\chi}_I (\chi_V \tilde{\chi}_C + \chi_C \tilde{\chi}_V) - \bar{\chi}_C (\chi_I \tilde{\chi}_S + \chi_S \tilde{\chi}_I)] \} \end{aligned} \quad (\text{A.9})$$

At this stage, either  $Z_A$ ,  $Z_B$ , or  $Z_C$  (or others) could potentially describe the finite-temperature behavior of the ten-dimensional  $SO(32)$  heterotic string. (For example, in Ref. [33] a different partition function is constructed and analyzed.) However, it is important to stress that these are not merely random modular-invariant functions which happen to exhibit mathematical interpolations away from the  $T = 0$  supersymmetric  $SO(32)$  endpoint. Rather, these functions actually emerge as the partition functions of *bona-fide nine-dimensional string models with an identifiable thermal radius of compactification*. In other words, they are the partition functions of nine-dimensional models which are explicitly constructed by compactifying the original zero-temperature model on a circle, and then implementing the  $\mathbb{Z}_2$  orbifold  $TQ$ . Only this can guarantee their internal self-consistency at the level of an appropriate worldsheet string construction. Indeed, the only differences between these possibilities correspond to the internal gauge-group Wilson lines, represented by the differences in the orbifold factors  $Q$ .

Given these candidate possibilities  $Z_A$ ,  $Z_B$ , and  $Z_C$ , certain features are already obvious. First, we observe that the CFT ground state  $\bar{\chi}_I \chi_I \chi_I$  or  $\bar{\chi}_I \chi_I \tilde{\chi}_I$  never appears in  $Z^{(4)}$  (*i.e.*, multiplying  $\mathcal{O}_{1/2}$ ). As discussed in Sect. 4.2, this is because the  $\mathcal{O}_{1/2}$  sector is a twisted sector, while the CFT ground state is necessarily untwisted. Indeed, this possibility is ruled by worldsheet self-consistency constraints stemming from the nature of the  $\mathbb{Z}_2$  thermal orbifold. We conclude, then, that this CFT ground state is always GSO-projected out of the spectrum, which in turn implies that none of these potential finite-temperature models would experience the traditional heterotic Hagedorn phase transition at  $a = 2 - \sqrt{2}$ . As discussed in Sect. 4.2, we believe that this is a general feature for all finite-temperature heterotic string models.

Another important feature is that each of these potential candidate functions contains the proto-graviton state multiplying  $\mathcal{E}_0$  and a proto-gravitino state multiplying  $\mathcal{E}_{1/2}$ . As a gauge singlet, the proto-graviton state is encoded within the contributions  $\bar{\chi}_V \chi_I^2$  or  $\bar{\chi}_V \chi_I \tilde{\chi}_I$ , while the proto-gravitino state is encoded within the contributions  $\bar{\chi}_S \chi_I^2$  or  $\bar{\chi}_S \chi_I \tilde{\chi}_I$ . Thus, as discussed in Sect. 5, each of these potential finite-temperature extensions of the  $SO(32)$  string would give rise to a re-identified, tenth-order Hagedorn transition at  $a = 2$ . Of course, as discussed in Sect. 6, this phase transition may become irrelevant if there are other Hagedorn-like phase transitions which emerge at lower temperatures.

Finally, we need to determine which of these candidate functions actually represents the finite-temperature behavior of the  $SO(32)$  heterotic string. Our selection is guided by following criterion. Clearly, each of the above partition functions represents a valid nine-dimensional model with an identifiable *geometric* radius of compactification. However, we can interpret this radius as a true *thermal* radius (*i.e.*, as a *temperature*) only if we impose the additional constraint that all massless states that appear multiplied by  $\mathcal{E}_0$  (for which the thermal excitations are periodic around the thermal circle) be spacetime bosons, while all massless states that appear multiplied by  $\mathcal{E}_{1/2}$  (for which the thermal excitations are antiperiodic around the thermal circle) be spacetime fermions. Note, in particular, that we do *not* make any demands on the  $\mathcal{O}_0$  or  $\mathcal{O}_{1/2}$  sectors, since these sectors necessarily have non-zero thermal winding modes. Such stringy states therefore have no field-theoretic limits, and are beyond our usual expectations. Likewise, by restricting our attention to only the *massless* states which multiply  $\mathcal{E}_0$  and  $\mathcal{E}_{1/2}$ , we are again focusing on only the light states which can emerge in an appropriate low-energy field-theoretic limit. We shall discuss the role of the Planck-scale states shortly.

Given this additional constraint, we immediately see that only  $Z_A$  in Eq. (A.5) has the required properties. Indeed, recalling the conformal dimensions listed above Eq. (A.3), we see that terms of the form  $\bar{\chi}_V \chi_V^2$  and  $\bar{\chi}_S \chi_V^2$  (or  $\bar{\chi}_V \chi_V \tilde{\chi}_V$  and  $\bar{\chi}_S \chi_V \tilde{\chi}_V$ ) contain contributions from massless spacetime bosons and fermions respectively, yet these terms appear in  $Z_A$  and  $Z_C$  multiplied by  $\mathcal{E}_{1/2}$  and  $\mathcal{E}_0$  respectively. Such massless “field-theoretic” states therefore violate finite-temperature spin-statistics, causing us to reject these possibilities. Thus  $Z_A$  is the unique self-consistent interpolating model which can represent the supersymmetric  $SO(32)$  heterotic string at finite temperature.

Given this result, we might be tempted to conclude that the supersymmetric  $SO(32)$  heterotic string gives rise to a re-identified Hagedorn transition at temperature  $a = 2$  stemming from the proto-gravitino thermal term within  $\bar{\chi}_S \chi_I^2 \mathcal{E}_{1/2}$  in  $Z_A$ . However, we notice that the partition function  $Z_A$  also contains  $(H_R, H_L) = (-1/2, -1/2)$  tachyons arising from the terms  $\bar{\chi}_I (\chi_I \chi_V + \chi_V \chi_I)$  multiplying  $\mathcal{O}_0$ . This is to be expected, since these are nothing but the 32 tachyons of the non-supersymmetric ten-dimensional  $SO(32)$  heterotic string that emerges in the  $T \rightarrow \infty$  limit. As discussed in Sect. 6, these tachyonic states give rise to a Hagedorn transition

at  $a = 1/\sqrt{2}$ , and this supplants the much weaker tenth-order Hagedorn transition that would have been induced at higher temperatures by the proto-gravitino. Thus, we conclude that the supersymmetric  $SO(32)$  heterotic string actually has a Hagedorn temperature  $a = 1/\sqrt{2}$ , just as for the Type IIA and Type IIB strings. In both cases, this behavior is induced by a  $(H_R, H_L) = (-1/2, -1/2)$  tachyon which emerges for  $a > 1/\sqrt{2}$ .

Having explicitly performed this analysis for the supersymmetric  $SO(32)$  heterotic string, we can quickly consider the cases of the ten-dimensional  $E_8 \times E_8$  and  $SO(16) \times SO(16)$  heterotic strings. The analysis is completely similar. In each case, we find that we can build many nine-dimensional interpolating models away from these zero-temperature limits, but in each case, we find that the self-consistent  $T \rightarrow \infty$  endpoint must be a  $D = 10$  tachyonic heterotic string model. Since the tachyons in these models are always at the same energy  $(H_R, H_L) = (-1/2, -1/2)$ , and since they all must arise within the  $\mathcal{O}_0$  sector of the interpolation, each of these models must also have a Hagedorn transition at  $a = 1/\sqrt{2}$ .

We conclude, then, that *all tachyon-free closed string models in ten dimensions share a universal Hagedorn temperature*. This applies not only to the Type IIA and IIB strings, but also to the supersymmetric  $SO(32)$ ,  $E_8 \times E_8$ , and non-supersymmetric  $SO(16) \times SO(16)$  heterotic strings as well. Even though the heterotic strings have a different CFT ground state than their Type II cousins, and even though their asymptotic densities of states rise more rapidly, their thermal GSO projections eliminate their corresponding (traditional) Hagedorn divergences at  $a = 2 - \sqrt{2} \approx 0.59$ , leaving behind one of the re-identified Hagedorn divergences at  $a = 1/\sqrt{2} \approx 0.71$ . These GSO projections thus restore a certain symmetry and universality between the Type II and heterotic strings, at least in ten dimensions, and suggest that perhaps a similar universality might exist in lower dimensions as well.

Our analysis has also yielded another surprise. It was, perhaps, already expected from Ref. [8] that states with non-trivial thermal winding modes might behave in a counter-intuitive fashion, violating finite-temperature spin-statistics relations in the  $\mathcal{O}_0$  and  $\mathcal{O}_{1/2}$  sectors. What is more surprising, however, is that all of our partition-function expressions necessarily have apparent thermal spin-statistics violations even for the states with *zero* windings, *i.e.*, states which appear in the  $\mathcal{E}_0$  and  $\mathcal{E}_{1/2}$  sectors. Fortunately, all of these violations are safely at the Planck scale; indeed, this was one of the criteria that we employed when selecting self-consistent interpolations. However, we now see quite generally that *Planck-scale* spin-statistics violations of this sort appear to be unavoidable, even for zero-winding states; they are required, in some sense, by modular invariance. It would be interesting to understand the thermal implications of these states as far as Planck-scale physics is concerned.

Finally, we stress again that it is important to actually construct such nine-dimensional interpolating models realized through *bona-fide* orbifolds and compactifications of our ten-dimensional models. Only this can guarantee the internal self-consistency of the resulting partition function, and the existence of an associated

worldsheet formulation. Otherwise, if our task were merely to construct modular-invariant expressions that appear to be mathematical finite-temperature interpolations, other possibilities would immediately arise. For example, if modular invariance were our only criterion, we could have begun again with our supersymmetric  $SO(32)$  heterotic string and constructed a trivial “interpolation” of the form

$$Z = Z_{\text{boson}}^{(7)} \times \{ \begin{array}{ll} \mathcal{E}_0 & \bar{\chi}_V (\chi_I^2 + \chi_V^2 + \chi_S^2 + \chi_C^2) \\ - \mathcal{E}_{1/2} & \bar{\chi}_S (\chi_I^2 + \chi_V^2 + \chi_S^2 + \chi_C^2) \\ - \mathcal{O}_0 & \bar{\chi}_C (\chi_I^2 + \chi_V^2 + \chi_S^2 + \chi_C^2) \\ + \mathcal{O}_{1/2} & \bar{\chi}_I (\chi_I^2 + \chi_V^2 + \chi_S^2 + \chi_C^2) \end{array} \}. \quad (\text{A.10})$$

Indeed, this interpolation would appear to have a traditional Hagedorn divergence stemming from the  $\bar{\chi}_I \chi_I^2 \mathcal{O}_{1/2}$  term, and would also apparently avoid thermal spin-statistics violations at the Planck scale by cleanly separating the zero-temperature bosonic and fermionic contributions within Eq. (A.1) into separate  $\mathcal{E}_0$  and  $\mathcal{E}_{1/2}$  sectors. However, it is easy to see that Eq. (A.10) cannot correspond to a *bona-fide* nine-dimensional string model. For example, Eq. (A.10) would appear to represent a non-supersymmetric interpolation between two *supersymmetric* limits, one at  $T = 0$  and the other at  $T \rightarrow \infty$ , both of which represent the same  $SO(32)$  heterotic string model but with opposite spacetime chiralities! Unfortunately, there is no self-consistent  $\mathbb{Z}_2$  orbifold  $Q$  which can accomplish this feat in ten dimensions. This example thus illustrates the need to have a proper worldsheet formulation for our partition function expressions before forming conclusions about their thermal behaviors. In other words, string consistency appears to *require* Planck-scale thermal spin-statistics violations, and in so doing eliminates the usual Hagedorn transition, replacing it with a re-identified Hagedorn transition at higher temperature.

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